ОБЛАСТИ СХОДИМОСТИ РЯДА ТЕЙЛОРА

Х. Г. Хаитова –

Бухарский государственный университет, преподаватель кафедры «Математический анализ»

Email: xilola_xayitova@mail.ru,

x.g.xayitova@buxdu.uz.

Аннотация. *В этой статье представлены некоторые концепции разложения функций в ряд Тейлора при некотором значении. Кроме того, представлены сходимости рядами Тейлора и анализ области сходимости. Здесь мы опишем некоторые из простейших примеров областей в пространстве C^n. Как обычно, область - это открытое связное множество, где открытость означает, что вместе с любой его точкой множество также содержит окрестность этой точки, а связность открытого множества D означает, что для любых точек z', z» $\in C^n$ существует непрерывное множество*

$$\gamma: [0,1] \to d$$

для которого

$$\gamma(0) = z' \text{ и } \gamma(1) = z''.$$

Ключевые слова. *Степенный ряды, комплексные числа, несколько переменных, точка, функция.*

THE DOMAINS OF CONVERGENCE OF THE TAYLOR SERIES

Kh.G.Khayitova –

teacher of the «Mathematical Analysis» department, Bukhara State University,

xilola_xayitova@mail.ru,

x.g.xayitova@buxdu.uz.

Annotation. *This article presents some concepts of Taylor series expansion of functions for some value. In addition, Taylor series convergence and convergence domain analysis are presented. Here we shall describe some of the simplest examples of domains in the space C^n. As usual, a domain is an open connected set, where openness means that along with any point of it the set also contains a neighborhood of that point, and connectedness of an open set D means that, for any points $z', z'' \epsilon D$ there exists a continuous are*

$$\gamma: [0,1] \to d$$

for which

$$\gamma(0) = z' \text{ and } \gamma(1) = z''.$$

Key words: *degree series, several variables, complex numbers, point, function.*

One of the main theorems of the theory of a complex variable is

Theorem: *Let $f \epsilon O(D)$ and let $z0 \epsilon D$ be an arbitrary point in D. Then the function f may be represented as a sum of a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

inside any disk $U = \{|z - z_0| < R\} \subset D$.

Proof. Let $z \epsilon U$ be an arbitrary point. Choose $r > 0$ so that $|z - z\_0| < r < R$ and denote by $\{\gamma_r = \varsigma : |\varsigma - z| = r\}$. The integral Cauchy formula implies that

$$f(z) = \frac{1}{2\pi i} \int_D \frac{f(\varsigma)}{\varsigma - z} d\varsigma.$$

in order to represent f as a power series let us represent the kernel of this integral as the sum of a geometric series:

$$\frac{1}{\varsigma - z} = \left[(\varsigma - z_0)\left(1 - \frac{z - z_0}{\varsigma - z_0}\right)\right]^{-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\varsigma - z_0)^{n+1}}.$$

We multiply both sides by $\frac{1}{2\pi i}f(\varsigma)$ and integrate the series term-wise along γ_r. The series converges uniformly on γ_r since

$$\left|\frac{z-z_0}{\varsigma-z_0}\right| = \frac{|z-z_0|}{r} = q < 1$$

for all $\varsigma\epsilon\gamma_r$. Uniform convergence is preserved under multiplication by a continuous and hence bounded function $\frac{1}{2\pi i}f(\varsigma)$. Therefore our term-wise integration is legitimate and we obtain [1-7]

$$f(z) = \frac{1}{2\pi i}\int_D \frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}}(z-z_0)^n = \sum_{n=0}^{\infty} c_n\,(z-z_0)^n$$

where

$$c_n = \frac{1}{2\pi i}\int_D \frac{f(\varsigma)d\varsigma}{(\varsigma-z_0)^{n+1}} \qquad n = 0,1,\dots$$

Definition: *The power series with coefficient given by is the Taylor series of the function f at the point z_0 (or centered at z_0).*

The Cauchy theorem implies that the coefficients c_n of the Taylor series defined by do not depend on the radius r of the circle γ_r, $0 < r < R$.

Exercise 1. Find the radius of the largest disk where the function $z\ /\ sin\ z$ may be represented by a Taylor series centered at $z_0 = 0$.

Exercice 2. Let f be holomorphic in C. Show that (a) f is even if and only if its Taylor series at $z = 0$ contains only even powers; (b) f is real on the real axis if and only if $f(\bar{z}) = \overline{f(z)}$ for all $z \in C$. We present some simple corollaries of Theorem [7-13].

The Cauchy inequalities. Let the function f be holomorphic on a closed disk $\overline{\text{U}} = \{|z-z_0| \le r\}$ and let its absolute value on the circle $\gamma_r = \partial\text{U}$ be bounded by a constant M. then the coefficients of the Taylor series of f at z_0 satisfy the inequalities

$$|c| \le \text{M}/r^n \qquad (n = 0,1,\dots)$$

Proof. We deduce from exercise 1 using the fact that $|f(\varsigma)| \leq \mathrm{M}$ for all $\varsigma\epsilon\gamma_r$:

$$|c_n| \leq \frac{1}{2\pi}\frac{\mathrm{M}}{r^{n+1}}2\pi r = \frac{\mathrm{M}}{r^n}.$$

Exercise 3. Let $P(z)$ be a polynomial in z of degree n. Show that if $|p(z)| \leq \mathrm{M}$ for $|z| = 1$ then $|p(z)| \leq \mathrm{m}|z|^{\mathrm{n}}$ for all $|z| \geq 1$.

The Cauchy inequalities simply the interesting.

Theorem . *(Liouville) If the function f is holomorphic in the whole complex plane and bounded then it is equal identically to a constant.*

Proof. According to theorem the function f may be represented by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in any closed disk $\overline{\mathrm{U}} = \{|z| \leq r\}$, $r < \infty$ with the coefficients that do not depend on R. Since f is bounded in C, say $|f(z)| \leq M$ then the Cauchy inequalities imply that for any $n = 0,1,....$ we have $|c_n| \leq \mathrm{M}/\mathrm{R}^n$. We may take R to be arbitrary large and hence the right side tends to zero as $R \to +\infty$ while the left side is independent of R. Therefore the two properties of a function-to be holomorphic and bounded are realized simultaneously only for the trivial functions that are equal identically to a constant [10-25].

Theorem: *If a function f is holomorphic in the closed complex plane C then it is equal identically to a constant*.

Proof. If the function f is holomorphic at infinity the limit $\lim_{z\to\infty} f(z)$ exists and is finite. Therefore f is bounded in a neighborhood $U = \{|z| > r\}$ of this point. However, f is also bounded in the complement $\mathrm{U}^c = \{|z| \leq r\}$ since it is continuous there and the set U^c is compact. Therefore f is holomorphic and bounded in C and thus Theorem implies that is equal to a constant.

Theorem: *Claims that any function holomorphic in a disk may be represented as a sum of a convergent power series inside this disk. We would like to show now that, conversely, the sum of a convergent power series is a holomorphic function. Let us first recall some properties of power series that are familiar from the real analysis.*

Lemma: *If the terms of a power series*

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

are bounded at some point $z_0 \epsilon c$, that is

$$|c_n (z_0 - a)^n| \leq M, \quad (n = 0,1,2 \dots)$$

Then the series converges in the disk $U = \{z : |z-a| < |z_0 - a|\}$. Moreover, it converges absolutely and uniformly on any set K that is properly contained in U.

Proof. We may assume that $z_0 \neq a$, so that $|z_0 - a| = \rho > 0$, otherwise the set U is empty. Let K be properly contained in U, then there exists $q < 1$ so that $|z-a|/\rho \leq q < 1$ for all $z \in U$. Therefore for any $z \in K$ and any $n \in N$ we have

$|c^n (z_0 - a)| \leq |c|\rho^n p^n$. However, assumption implies that $|c|\rho^n \leq M$ so that the series is majorized by a convergent series $M \sum_{n=0}^{\infty} q_n$ for all $z \in K$. Therefore the series converges uniformly and absolutely on K. This proves the second statement of this lemma. The first one follows from the second since any point $z \in U$ belongs to a disk $\{|z-a| < \rho'\}$ with $\rho' < \rho$, that is properly contained in U.

Theorem*: (Abel) Let the power converge at a point $z_0 \in C$. Then this series converges in the disk $U = \{z : |z-a| < |z_0 - a|\}$ and, moreover, it converges uniformly and absolutely on every compact subset of U.*

Proof. Since the series converges at a point z_0 the terms $c_n (z_0 - a)^n$ converge to zero as $n \to \infty$. However, every converging sequence is bounded, and hence the assumptions of the previous lemma are satisfied both claims of the present theorem follow from this lemma [12-25].

The Cauchy – Hadamard formula. *Let the coefficients of the power series satisty*

$$\lim_{n\to\infty} sup|c_n|^{1/n} = \frac{1}{\mathrm{R}},$$

with $0 \le r \le \infty$. *Then the series*

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

converges at all z such that $|z-a| < R$ *and diverges at all z such that*

$$|z-a| > R.$$

Proof. Recall that $A = \lim_{n\to\infty} sup\alpha_n$ if there exists a subsequence $\alpha_{n_k} \to \mathrm{A}$ as $k \to \infty$, and for any $\varepsilon > 0$ there exists $n \in N$ so that $\alpha_n < a + \varepsilon$ for all $n \ge \mathrm{N}$. This includes the cases $A = \pm\infty$. However, if $A = +\infty$ then condition is not necessary, and if $A = -\infty$ then the number $A + \varepsilon$ in condition is replaced by an arbitrary number. It is shown in real analysis that $\lim_{n\to\infty} sup\alpha_n$ exists for any sequence $\alpha_n \in r$.

Let $0 < r < \infty$, then for any $\varepsilon > 0$ we may find N such that for all $n \ge N$ we have $|c_n|^{1/n} \le \frac{1}{\mathrm{R}} + \varepsilon$. Therefore, we have [11-25]

$$|c_n(z-a)^n| < \left\{\left(\frac{1}{\mathrm{R}} + \varepsilon\right)|z-a|\right\}^n.$$

Furthermore, given $z \in C$ such that $|z-a| < R$ we may choose ε so small that we have $\left(\frac{1}{R} + \varepsilon\right)|z-a| = q < 1$. Then shows that the terms of the series are majorized by a convergent geometric series q^n for $n \ge N$, and hence the series converges when $|z-a| < R$.

Condition in the definition of $\lim_{n\to\infty} sup\alpha_n$ implies that for any $\varepsilon > 0$ one may find a subsequence c_{n_k} so that $\left|c_{n_k}\right|^{1/n_k} > \frac{1}{\mathrm{R}} - \varepsilon$ and hence

$$\left|c_{n_k}(z-a)^{n_k}\right| > \left\{\left(\frac{1}{\mathrm{R}} - \varepsilon\right)|z-a|\right\}^{n_k}.$$

Then, given $z \in C$ such that $|z-a| > R$ we may choose ε so small that we have $\left(\frac{1}{R} - \varepsilon\right)|z-a| > 1$. Then implies that $\left|c_{n_k}(z-a)^{n_k}\right| > 1$ for all k and hence the n −th term of the power series does not vanish as $n \to \infty$ so that the series diverges $|z-a| > R$.

We leave the proof in the special case $R = 0$ and $R = \infty$ as an exercise for the reader.

Definition. The domain of convergence of a power series is the interior of the set E of the points $z \in C$ where the series converges.

REFERENCES

1. Xayitova X. Chiziqli tenglamalarni o'qitishda «Aqliy hujum» va «kichik guruhlarda ishlash» metodlaridan foydalanish //Центр научных публикаций (buxdu. uz). – 2021. – Т. 8. – №. 8.
2. Gafurovna K. K. Some applications of the derivative of a function //Лучшие интеллектуальные исследования. – 2024. – Т. 19. – №. 3. – С. 7-12.
3. G'afurovna X. X. et al. Olmos panjaradagi diskret Shryodinger operatorining spektri //Journal of new century innovations. – 2023. Т. 29. – №. 2. – С. 120-125.
4. Xayitova X. Funksional tenglamalar yechishning ayrim metodlari //Центр научных публикаций (buxdu. uz). – 2024. – Т. 45. – №. 45.
5. Xayitova X. Tanlash usuli bilan kombinatorika masalalarni yechish metodikasi: tanlash usuli bilan kombinatorika masalalarni yechish metodikasi //Центр научных публикаций (buxdu. uz). – 2022. – Т. 25. – №. 25.
6. Хайитова Х.Г. Использование эвристического метода при объяснении темы «Непрерывные линейные операторы» по предмету «Функциональный анализ» // Вестник науки и образования. 94:16, 2020. Часть 2. С. 25-28.
7. Хайитова Х.Г. Преимущества использовании метода анализа при изучении темы «Непрерывные функции» по предмету «Математический анализ» // Проблемы педагогики, 2021 № 2(53). С. 46-49.

8. Xayitova X.G., Ramazonova Sh.Sh., Panjaradagi ikki o'lchamli qo'zg'alishga ega bilaplasian operatorining spektri va rezolventasi. Science and education. Vol. 3 No. 3 (2022), 55-64.

9. Хайитова Х.Г. О числе собственных значений модели Фридрихса с двухмерным возмущением. Наука, техника и образование. 2020. 8 (72), 5-8.

10. Хайитова Х.Г., «Преимущества использования метода научного исследования при решении задач комбинаторики» Научный импульс №10(100). Часть 2, Москва 2023 г.

11. U.U.Umarova Application of TRIZ technology to the topic «Normal forms for formulas of the algebra of statements» // Science, technology and education. 73: 9 (2020), pp. 32-35.

12. Umarova U.U. The method of «Working in small groups» in teaching the topic of logical operations on feedback // Scientific progress, 2: 6 (2021), p. 803-809.

13. Umarova U.U. Problem-based learning technology in finding a formula using a truth table // Scientific progress, 2: 6 (2021), p. 832-838.

14. Umarova U.U. Graphic organizer methods in the repetition of the section of feedback algebra // Scientific progress, 2: 6 (2021), p. 825-831.

15. Umarova U.U. «Brainstorming» and «Sase Study» methods in teaching the topic «Basic equally powerful formulas of reasoning algebra» // Scientific progress, 2: 6 (2021), p. 818-824.

16. Умарова У.У. «Функциялар системасининг тўлиқлиги ва ёпиқлиги» мавзусини ўқитишда «Қандай?» иерархик диаграммаси интерфаол методи // Scientific progress, **2**:6 (2021), p. 855-860

17. Умарова У.У. «Мулоҳазалар хисоби» мавзусини ўқитишда интерфаол методлар // Scientific progress, **2**:6 (2021), p. 867-875

18. Umarova U.U. «Equivalence of formulas. The principle of duality «in the teaching of» Charkhpalak «technology // Scientific progress, 2:6 (2021), p. 839-846.

19. Umarova U.U., Ikromova S.U. Umumiy oʻrta taʼlim maktablarida masofaviy o ʻqitish // Science and Education 2 (9), 374-382.

20. Umarova U.U. Problem-based learning technology in finding a formula using a truth table // Scientific progress, 2: 6 (2021), p. 832-838.

21. Umarova U.U. Interactive methods in teaching the topic of «Accounting for feedback» // Scientific progress, 2: 6 (2021), p. 867-875.

22. Umarova U.U. «Relationships. Binary Relationships» and «Dominoes» methods for lectures and practical classes // Scientific progress, 2: 6 (2021), p. 982-988.

23. Умарова У.У. «Келтириб чиқариш қоидаси» мавзусини ўқитишда график органайзер методлар // Scientific progress, **2**:6 (2021), p. 876-882.

24. Umarova U.U. «Blitz-survey» and «FSMU» technology in a practical lesson on «Post theorem and its results» // Scientific progress, 2: 6 (2021), p. 861-866.

25. Umarova U.U. Boomerang technology in teaching the topic «Primitive recursive functions» // Scientific progress, 2: 6 (2021), p. 890-897.