ON THE CONVERGENCE DOMAIN OF COMPLEX SERIES

Khayitova Khilola Gafurovna,

Teacher of the Department of Mathematical Analysis, Bukhara State University

Annotation. *This article studies power series with complex variables and their convergence domain. Some properties of converging series are presented. The radius of convergence of a power series is studied.*

Key words: *variable, radius of convergence, function, holomorph, Reinhardt domains.*

ОБ ОБЛАСТИ СХОДИМОСТИ КОМПЛЕКСНЫХ РЯДОВ

Аннотация. *В статье изучаются степенные ряды с комплексными переменными и область их сходимости. Приведены некоторые свойства сходящихся рядов. Изучен радиус подхода горизонтального ряда.*

Ключевые слова: *переменная, радиус сходимости, функция, голоморф, домены Рейнхардта.*

Lemma. *If the function f is holomorphic with respect to each variable z_v in the polydisc*

$$u = u(a, r)$$

and bounded in U, then it is continuous at each point of U with respect to the set of all the variables.

Proof. Let $z^0, z \in U$ be arbitrary points. We write the increment of f as a sum of the increments with respect to the individual coordinates

$$f(z)-f(z^0)==\sum_{v=1}^{n}\{f(z_1^0,\dots,z_{v-1}^0,z_v,\dots,z_n)-f(z_1^0,\dots,z_v^0,z_{v+1},\dots,z_n)\}$$

and consider the *v*th summand as a function φ_v of the variable z_v with the remaining values of the argument being fixed. If $|f|\le \mathrm{M}/2$ in *U*, then the function φ_v satisfies the hypotheses of Schwarz's lemma in the form just given, and, applying inequality to each term of the sum $\left(f(z)-f(z^0)\right)$, we will find that

$$\left|\left(f(z)-f(z^0)\right)\right|\le \mathrm{M}\sum_{v=1}^{n} r_v\frac{|z_v-z_v^0|}{\left|r_v^2-\overline{z_v^0}z_v\right|}.$$

The assertion follows from this.

Thus, to prove Hartogs's theorem it remains to prove the bounded in some polydisc with center at *a* of a function that is holomorphic with respect to each variable. We not that the bounded in some polydisc, not necessarily with center at *a*, follows solely from the continuity of f in the separate variables. This fact forms the content of what is called lemma.

Lemma. *We represent the polydisc*

$$U=\{z\in \mathrm{C}^n:\ ||z||<R\}$$

as a product of

$$U=\{z'\in C^{n-1}: ||z'||<R\}$$

by the disc

$$U_n=\{z_n\in C:|z_n|<R\}.$$

If the function $f(z',z_n)$ *is continuous with respect to* $z'\in\overline{U}$, *then there exists a polydisc*

$$w=w'\times u_n\subset U$$

in which f *is bounded.*

Proof. For fixed $z\in\overline{U}$ we write

$$\mathrm{M}(z')=max|f(z',z_n)|$$

and consider the sets

$$\mathrm{E}_m = \{z' \in \overline{\mathrm{U}} : \mathrm{M}(z') \le m\}.$$

These sets are closed, since if

$$z^{(\mu)} \in E_m, \ (\mu = 1,2, \dots) \ \text{ and } \ z^{(\mu)} \to z,$$

then $z' \in E_m$. Obviously, the E_m from an increasing sequence and any point $z' \epsilon \overline{\mathrm{U}}$ belongs to all the E_m beginning with one of them. There exists an E_m containing some domain $\mathrm{g}' \subset \mathrm{U}'$. In fact, otherwise all the $\mathrm{E}_m$would be nowhere dense, but then in U' there would exits a ball $\overline{\mathrm{B}}^2$ not containing any points of E_2, and so on. Thus, there exists a domain G' in which

$$|f(z', z_n)| \le \mathrm{M}$$

for any $z_n \in \mathrm{U}_n$. It remains to choose a polydisc

$$w' = \{z' \big| |z' - z^0| \big| < r\}$$

in G', and then we will have

$$|f| \le \mathrm{M} \quad \text{in} \quad w = w' \times \mathrm{U}_n.$$

Lemma. *If the function* $f(z', z_n)$ *is holomorphic with respect to* z' *in* $\bar{V}$ *for any* $z_n \in \overline{U}_n$ *and holomorphic with respect to z in* $\overline{w}$*, then it is holomorphic in the entire polydisc* $\bar{V}$*.*

Proof. Without loss of generality we may assume $a' = 0$. For any fixed $z_n \in \mathrm{U}_n$ and any $z' \in \mathrm{V}'$ the function f is represented with respect to z' by the convergent power series

$$f(z) = \sum_{|k|=0}^{\infty} c_k(z_n)(z)^k,$$

where

$$k = (k_1, \dots, k_{n-1}).$$

The coefficients of this series

$$c_k(z_n) = \frac{1}{k!} \frac{\partial^{|k|} f(0, z_n)}{\partial' z^k}$$

are holomorphic in the disc U_n as derivatives of a function that is holomorphic in z_n. Therefore the function

$$\frac{1}{|k|}\ln|c_k(z_n)|$$

are subharmonic in U_n.

We choose an arbitrary number $\rho < \mathrm{R}$ since for any $z_n \in \mathrm{U}_n$

$$|c_k(z_n)|\rho^{|k|} \to 0$$

as $|k| \to \infty$, then for any $z_n \in \mathrm{U}_n$ there is a $|k|$ beginning with which we will have

$$\frac{1}{|k|}\ln|c_k(z_n)| + \ln\rho \le 0, \text{ i.e.,}$$

$$\overline{\lim_{|k|\to\infty}} \frac{1}{|k|}\ln|c_k(z_n)| \le \ln\frac{1}{\rho}.$$

Now we use the holomorphy of f in $\overline{\mathrm{W}}$; f is bounded in $\overline{\mathrm{W}}$, and the Cauchy inequalities

$$|c_k(z_n)|r^{|k|} \le \mathrm{M}$$

hold for any $z_n \in \mathrm{U}_n$. Therefore for any $z_n \in \mathrm{U}_n$ and any $|k|$

$$\frac{1}{|k|}\ln|c_k(z_n)| \le \mathrm{Ln}\frac{\mathrm{M}^{1/|k|}}{r} \le \mathrm{A}.$$

Thus these subharmonic functions satisfy the hypotheses of the lim-sup theorem. By this theorem for any $\omega < \rho$ one can find a number k_0 such that for all

$$|k| > k_0 \quad \text{and all} \quad z_n, |z_n| \le \omega,$$

we have

$$\frac{1}{|k|}\ln|c_k(z_n)| \le \ln\frac{1}{\omega},$$

$$|c_k(z_n)|\omega^{|k|} \le 1.$$

From this is follows that the series converges uniformly in any polydisc

$$\overline{\mathrm{U}}(0,\omega'), \omega' < \omega;$$

but the terms of this series are continuous in z, so that the sum f of the series is also continuous, and hence is bounded in $\mathrm{U}(0,\omega')$. This polydisc can be assumed to be arbitrarily close to V, and since V from the very beginning could be increased a little, then f is bounded, it is holomorphic in $\overline{\mathrm{V}}$.

Hartog's Theorem. *If the function f is holomorphic at any point of the domain $D \subset C^n$ with respect to each of the variables z_v, then it is holomorphic in D.*

Proof. If suffices to prove the holomorphy of f at an arbitrary point $z_0 \in D$, and without loss of generality we may assume that $z_0 = 0$. Thus, suppose f is holomorphic with respect to each variable in the polydisc $\overline{\mathrm{U}(0,\mathrm{R})}$ we must prove that it is holomorphic in some polydisc with center at 0.

We shall prove this assertion by induction on the number of complex variables. For one variable it is trivial; assume that it is true for functions of $(n-1)$ variables, and we let

$$\mathrm{U} = \mathrm{U}(0, \mathrm{R}/3).$$

Now we consider the polydisc

$$\mathrm{V} = \mathrm{V}' \times \mathrm{U}_n,$$

where

$$\mathrm{V}' = \mathrm{U}\left(a, \frac{2}{3}\mathrm{R}\right).$$

Obviously,

$$\overline{\mathrm{V}} \subset \overline{\mathrm{U}(0,\mathrm{R})},$$

and hence f is holomorphic with respect to $'z$ in $\overline{\mathrm{V}}$ for any $z_n \in \overline{\mathrm{U}_n}$, and by what we just proved it is holomorphic with respect to z in $\overline{\mathrm{W}}$. By Hartog's lemma it follows from this that it is also holomorphic with respect to z in V, which already contains the point $z = 0$. Thus the assertion is proved also for functions of n variables.

Here we consider basic questions related to series expansions of holomorphic functions.

Power series. In subsection we proved that any function that is holomorphic in the polydisc $U(a,r)$ can be expanded in this disc in a multiple power series with center at a. The question series concerning the set of points of convergence of this series. By analogy with functions of one variable one wants to expect that this set will be the polydisc, completed by some set of boundary points. However, very simple examples show that the situation is quite different.

Definition. The domain of convergence of the power series

$$\sum_{|k|=0}^{\infty} c_k(z-a)^k$$

is the interior S' of the set S of points $z \in C^n$ at which this series converges for some ordering of its terms.

Theorem. *If the point z^0 belongs to the domain of convergence S of the series, then the closed polydisc*

$$\overline{U} = \{z \epsilon C^n : |z_v - a_v| \le |z_v^0 - a_v|\}$$

also belongs to S' and the series converges absolutely and uniformly in $\overline{U}$.

Proof. Since $z^0 \in S'$ and $S'$is open, there exists a point $\varsigma \in S'$ such that

$$|\varsigma_v - a_v| > |z_v^0 - a_v|, \quad v = 1,2, \dots, n,$$

and the series converges at this point. Since

$$U \subset\subset \{z \in C^n : |z_v - a_v| < |\varsigma_v - a_v|\},$$

then by Abel's lemma the series converges absolutely and uniformly in $\overline{U}$.

Theorem 1 can also be formulated as follows: the domain of convergence S' of the series is a complete Reinhardt domain with center at a. Thus, complete Reinhardt domains play the same role in the case of functions of several variables as disc do in the case of one variable. This analogy is stressed by the following theorem.

Theorem. *Any function f that is holomorphic in a complete Reinhardt domain $D \subset C^n$ with center at a is represented in this domain by the Taylor expansion*

$$f(z)=\sum_{|k|=0}^{\infty} c_k(z-a)^k.$$

Proof. Let z^0 be an arbitrary point of D. Then the polydisc

$$\overline{\mathrm{U}}=\{|z_v-a_v|\le|z_v^0-a_v|\}\subset\subset \mathrm{D}$$

and by Theorem 2 of the function f is represented in U by a Taylor expansion centered at a. The coefficients of this expansion are computed via the derivatives of f at a, and hence, coincide with c_k, i.e., this expansion is the same as theorem 1. A natural question arises: is every complete Reinhardt domain the domain of convergence of some power series [1-27].

Definition. We denote by

$$\mathrm{z}\to\lambda(\mathrm{z})=(\ln|\mathrm{z}_1|,\dots,\ln|\mathrm{z}_\mathrm{n}|)$$

a mapping of the set

$$\{\mathrm{z}\in\mathrm{C}^\mathrm{n}:\ \mathrm{z}_1\dots\mathrm{z}_\mathrm{n}\neq 0\}$$

into the space R^n. The logarithmic image of a set $\mathrm{M}\subset\mathrm{C}^\mathrm{n}$ is the set $\mathrm{M}^*=\lambda(\mathrm{M}_\mathrm{v})$. The set M is said to be logarithmically convex if its logarithmic image M^* is a convex set in R^n.

REFERENCES

1. Хайитова Х.Г. Использование эвристического метода при объяснении темы «Непрерывные линейные операторы» по предмету «Функциональный анализ» // Вестник науки и образования. 94:16 (2020), часть 2. С. 25-28.

2. Хайитова Х.Г., Рустамова Б.И. Метод обобщения при обучении математике в школе // Проблемы педагогики 51:6 (2020). С. 45-48.

3. X.G‘.Xayitova Oliy ta'lim muassasalarida "Funksional analiz" fanini o'qitishda muammoli ta'lim metodida foydalanish // Современная психология и педагогика: проблемы, анализ и результаты, 227-230.

4. K.Khayitova The domain of convergence of the double degree seriers of several variables of the complex numbers // Journal of Global Research in Mathematical Archives 6:11(2019), 55-57.

5. X.G‘.Xayitova O'rta maktab matematika kursida tub va murakkab sonlari o'qitishda taqqoslash metodidan foydalanish // Pedagogik mahorat. 5-son 2019-yil, 139-141.

6. X.G‘.Xayitova O'rta maktabda matematika fanini o'qitishda umumlashtirish metodining afzalliklari // Pedagogik mahorat. 5-son 2020-yil, 122-1241.

7. Хайитова Х.Г. Преимущества использовании метода анализа при изучении темы «Непрерывные функции» по предмету «Математический анализ» // Проблемы педагогики, 2021 № 2(53). С. 46-49.

8. Xayitova X.G., Ramazonova Sh.Sh., Panjaradagi ikki o’lchamli qo’zg’alishga ega bilaplasian operatorining spektri va rezolventasi. Science and education. Vol. 3 No. 3 (2022), 55-64.

9. Хайитова Х.Г., О числе собственных значений модели Фридрихса с двухмерным возмущением. Наука, техника и образование. 2020. № 8 (72), 5-8.

10. Хайитова Х.Г., «Преимущества использования метода научного исследования при решении задач комбинаторики» Научный импульс №10(100). Часть 2, Москва 2023 г.

11. Umarova U.U. Interactive methods in teaching the topic of “Accounting for feedback” // Scientific progress, 2: 6 (2021), p. 867-875.

12. Umarova U.U. “Equivalence of formulas. The principle of "duality" in the teaching of "Charkhpalak" technology // Scientific progress, 2: 6 (2021), p. 839-846.

13. Umarova U.U. “Blitz-survey” and “FSMU” technology in a practical lesson on “Post theorem and its results” // Scientific progress, 2: 6 (2021), p. 861-866.

14. Umarova U.U. "How?" hierarchical diagram interactive method // Scientific progress, 2: 6 (2021), p. 855-860.

15. Umarova U.U. Graphic organizer methods in teaching the topic "Rules of derivation" // Scientific progress, 2: 6 (2021), p. 876-882.

16. Umarova U.U., S.U. Ikromova. Distance learning in general secondary schools // Science and Education 2 (9), 374-382.

17. Умарова У.У., Отамуродов Ф.Р. Алгоритм работы с приёмом “Корзина идей” и применение к теме “Полином Жегалкина” // Наука, техника и образование. 77:2 (2021). С. 42-45.

18. Umarova U. U. Application of TRIZ technology to the topic "Normal forms for formulas of propositional algebra". Science, technology and education. 73: 9 (2020), pp. 32-35.

19. Umarova U.U. Graphic organizer methods in the repetition of the section of feedback algebra // Scientific progress, 2: 6 (2021), p. 825-831.

20. Umarova U.U. “Brainstorming” and “Sase Study” methods in teaching the topic “Basic equally powerful formulas of reasoning algebra” // Scientific progress, 2: 6 (2021), p. 818-824.

21. Umarova U.U. Problem-based learning technology in finding a formula using a truth table // Scientific progress, 2: 6 (2021), p. 832-838.

22. Umarova U.U. The method of "Working in small groups" in teaching the topic of logical operations on feedback // Scientific progress, 2: 6 (2021), p. 803-809.

23. Umarova U.U. “Relationships. “Binary Relationships” and “Domino” methods for lectures and practical classes // Scientific progress, 2: 6 (2021), p. 982-988.

24. Boboyeva M.N. Maktab matematika darslarida misol-masalalar yechish orqali turli kasblarga oid ma’lumotlarni singdirish. Science and Education 2:8 (2021), 496-504 b.

25. Boboyeva M.N. Differensial hisobning iqtisodda qo‘llanilishini takomillashtirish istiqbollari. Science and Education 2:8 (2021),476-485 b.

26. Boboyeva M.N. “Matritsalar haqida tushuncha va ular ustida amallar” mavzusini ayrim interfaol metodlardan foydalanib o‘qitish. Pedagogik mahorat Maxsus son (2021), 38-42 b.

27. Boboyeva M.N. Increasing creative activity of students by application of methods of analysis and synthesis in mathematics lessons. ResearchJet Journal of Analysis and Inventions. 3:05 (2022), p.67-75.