NOKORREKT VA TESKARI MASALALAR FANIDAN TO‘LQIN TENGLAMASI UCHUN DRIXLE MASALASINING KORREKTIV SHARTLARNING BUZILISHI

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Annotatsiya

Ushbu maqolamizda to‘lqin tenglamasi uchun qo‘yilgan Dirixle masalasining korrektivlik shartlari tahlil qilinib, ularning buzilishi oqibatida masalaning nokorrekt holatga o‘tishi o‘rganilgan. Tadqiqot davomida yechimning mavjudligi, yagonaligi va barqarorligi masalalari ko‘rib chiqilgan hamda energiya usuli yordamida yagonalik isbotlangan. Natijalar matematik modellashtirish va raqamli hisoblashlarda yechimning barqarorligini ta’minlash nuqtai nazaridan muhim ahamiyatga egaligi xususida so’z boradi.

Kalit so‘zlar:korrekt, matematik modellashtirish, raqamli hisoblash, to‘lqin tenglamasi, divergens, nokorrekt, xos son, xos funksiya.

Annotation

This scientific work analyzes the well-posedness conditions for the Dirichlet problem formulated for the wave equation and examines how the violation of these conditions leads to the problem becoming ill-posed. The study addresses the issues of existence, uniqueness, and stability of the solution, and proves uniqueness using the energy method. The results are of significant importance for ensuring the stability of solutions in mathematical modeling and numerical computations.

Keywords: well-posed, mathematical modeling, numerical computation, wave equation, divergence, ill-posed, eigenvalue, eigenfunction.

Аннотация

В данной научной работе проанализированы условия корректности для задачи Дирихле, поставленной для уравнения волны, и исследованы последствия нарушения этих условий, приводящие к переходу задачи в некорректную постановку. В ходе исследования рассмотрены вопросы существования, единственности и устойчивости решения, а также доказана единственность с использованием метода энергии. Полученные результаты имеют важное значение для обеспечения устойчивости решений в задачах математического моделирования и численных вычислений.

Ключевые слова: корректная постановка, математическое моделирование, численные вычисления, волновое уравнение, расходимость, некорректная постановка, собственное значение, собственная функция.

Kirish. To‘lqin tenglamasi matematik fizikaning asosiy tenglamalaridan biri bo‘lib, u turli fizikaviy jarayonlarni modellashtirishda keng qo‘llaniladi. Ushbu ishda aynan to‘lqin tenglamasi uchun qo‘yilgan Dirixle masalasining korrektiv shartlari va ularning buzilishi oqibatlari o‘rganiladi. Masalaning nokorrekt holga o‘tishi yechimning mavjud emasligi, yagonalikning yo‘qligi yoki barqarorlikning buzilishiga olib kelishi mumkin. Ayniqsa, chegaraviy shartlarning to‘liq bajarilishi, xususan Dirixle sharti, masalaning korrektivligi uchun muhim omil hisoblanadi. Shu nuqtai nazardan, ushbu tadqiqot to‘lqin jarayonlarini tahlil qilishda muhim nazariy va amaliy ahamiyatga ega.

Aytaylik, $D = \{0 < x < p, 0 < t < ap\}$ (x,t) tekisligida berilgan soha, bu yerda α - doimiy musbat son $U(x,t) \mathrm{O}(D)$ funksiyani to‘lqin tenglamasi uchun Dirixle masalasining yechimi deb ataymiz, agar quyidagi shartlar bajarilsa:

$$U_{tt} - a^2U_{xx} = 0 \qquad (1)$$

$$U_t(x,0) = j\ (x), U_t(x,T) = y\,(x), 0\,\mathrm{J}\ \ x\,\mathrm{J}\ \ T \qquad (2)$$

$$U(0,t) = U(l,t) = 0, 0\,\mathrm{J}\ \ t\,\mathrm{J}\ \ l \qquad (3)$$

bu yerda $j\ (x)$, $y\,(x)$ uzluksiz funksiyalar. (1)-(3) masala yechimining $\{j\ ,y\ ,a\}$ boshlang‘ich berilganlarga uzluksiz bog‘liqligi yo‘q. $U(x,t)$ funksiya topilsin.

Yechimni quyidagi ko‘rinishda izlaymiz:

Xos funksiyalarni topish uchun o'zgaruvchilarni ajratish usulini qo'llaymiz:

$$U(x,t) = X(x)T(t) \qquad (4)$$

Tenglamaga qo'yamiz:

$$X(x)T''(t) = a^2X''(x)T(t)$$

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\ l$$

Bu bizga ikkita oddiy differensial tenglama beradi.

1. Fazoviy qism:

$$X''(x) + l\ X(x) = 0 \qquad (5)$$

2. Vaqt qismi:

$$T''(t) + l\ a^2T(t) = 0 \qquad (6)$$

Fazoviy tenglama uchun xos qiymatlar masalasini yechamiz.

a) $l\ < 0$ holatida:

$$X(x) = Ae^{\sqrt{-\ l\ x}} + Be^{\sqrt{-\ l\ x}}$$

Chegaraviy shartlarni qo'llasak:

$$X(0) = A + B = 0$$

$$X(l) = Ae^{\sqrt{-\ l\ x}} + Be^{-\ \sqrt{-\ l\ x}} = 0$$

Bu faqat A=B=0 da bajariladi. Bu biz uchun ahamiyatsiz yechim.

b) $l\ = 0$ holatida:

$$X(x) = Ax + B$$

Chegaraviy shartlarni qo'llasak:

$$X(0) = B = 0$$

$$X(l) = Al = 0 \rightarrow A = 0$$

Bu yechim ham biz uchun kerak emas.

c) $\lambda > 0$ holatida:

$$X(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$$

Yana chegaraviy shartlarni qo‘llasak:

$$X(0) = A = 0$$

$$X(l) = B\sin(\sqrt{\lambda}\, l) = 0$$

Rtivial bo‘lmagan yechim uchun:

$$\sin(\sqrt{\lambda}\, l) = 0 \rightarrow \sqrt{\lambda}\, l = \pi k, k = 1,2,3,...$$

$$\lambda_k = \left(\frac{\pi k}{l}\right)^2$$

Xos sonni topib oldik. Endi esa Xos funksiyani topamiz.

$$X_k(x) = \sin\left(\frac{\pi k x}{l}\right), k = 1,2,3,...$$

Har bir λ_k uchun vaqt tenglamasini tuzamiz. λ_k ni olib borib (6) formulaga qo‘yamiz.

$$T_k''(t) + \left(\frac{a\pi k}{l}\right)^2 T_k(t) = 0 \tag{7}$$

Har bir λ_k uchun vaqt tenglamasini tuzganimizdan so‘ng umumiy tenglamasini ham tuzib olamiz, ya’ni (7) tenglikni (4) tenglikka olib borib qo‘yamiz:

$$T_k(t) = a_k \cos\sqrt{\lambda_k}\, at + b_k \sin\sqrt{\lambda_k}\, at,$$

$$U(x,t) = \sum_{k=1}^{\infty} \left[a_k \cos\sqrt{\lambda_k}\, at + b_k \sin\sqrt{\lambda_k}\, at\right] \sqrt{\frac{2}{l}} \sin\sqrt{\lambda_k}\, x, k \in N \tag{8}$$

Umumiy tenglamani tuzib olganimizdan keyin uni boshlang‘ich shartlarga qo‘yamiz . a_k va b_k larni topib olamiz.

$$U_t(x,t)=\sum_{k=1}^{\infty}\left[a_k\sqrt{\lambda_k}a\sin\sqrt{\lambda_k}at+b_k\sqrt{\lambda_k}a\cos\sqrt{\lambda_k}at\right]\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x,k\in N$$

$$U_t(x,0)=\sum_{k=1}^{\infty}\left[a_k\sqrt{\lambda_k}a*0+b_k\sqrt{\lambda_k}a*1\right]\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x=\varphi(x),k\in N,$$

$$U_t(x,T)=\sum_{k=1}^{\infty}\left[-a_k\sqrt{\lambda_k}a\sin\sqrt{\lambda_k}aT+b_k\sqrt{\lambda_k}a\cos\sqrt{\lambda_k}aT\right]\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x=\psi(x),k\in N$$

$$\varphi_k=\sqrt{\frac{2}{l}}\int_0^l\varphi(x)\sin\sqrt{\lambda_k}xdx\ \to\ \varphi(x)=\sum_{k=1}^{\infty}\sqrt{\frac{2}{l}}\varphi_k\sin\sqrt{\lambda_k}xdx,$$

$$\sum_{k=1}^{\infty}\left[-a_k*0+b_k*1\right]\sqrt{\lambda_k}a\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x=\sum_{k=1}^{\infty}\sqrt{\frac{2}{l}}\varphi_k\sin\sqrt{\lambda_k}x,$$

$$b_k=\frac{\varphi_k}{\sqrt{\lambda_k}a}=\frac{l}{a\pi k},\quad \varphi(x)=\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi x}{l} \tag{9}$$

$$U(x,t)=\sum_{k=1}^{\infty}\sin\frac{k\pi at}{l}\sin\frac{k\pi x}{l} \tag{10}$$

b_k ni topib oldik. Endi a_k ni b_k orqali topib olamiz.

$$\sum_{k=1}^{\infty}\left[a_k\cos\sqrt{\lambda_k}aT+b_k\sin\sqrt{\lambda_k}aT\right]\sqrt{\lambda_k}a\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x=\sum_{k=1}^{\infty}\sqrt{\frac{2}{l}}\psi_k\sin\sqrt{\lambda_k}x,$$

$$a_k=\frac{\psi_k}{\cos\sqrt{\lambda_k}aT}-\frac{\varphi_k}{\sqrt{\lambda_k}a}tg\sqrt{\lambda_k}aT. \tag{11}$$

a_k va b_k larni topib oldik. Endi (9) va (11) tengliklarni olib borib U(x,t) funksiyaga qo‘yamiz.

$$U(x,t)=\sum_{k=1}^{\infty}\left[\left(\frac{\psi_k}{\cos\sqrt{\lambda_k}aT}-\frac{\varphi_k}{\sqrt{\lambda_k}a}tg\sqrt{\lambda_k}aT\right)\sqrt{\lambda_k}a\sin\sqrt{\lambda_k}at+\left(\frac{\varphi_k}{\sqrt{\lambda_k}a}\right)\sqrt{\lambda_k}a\cos\sqrt{\lambda_k}at\right]\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x$$

Yuqoridagi U(x,t) funksiyani soddalashtirib qo‘yamiz.

$$U_k(x,t)=\sum_{k=1}^{\infty}\psi_k\sqrt{\frac{2}{l}}\sin\sqrt{\lambda_k}x\ \to\ U(x,t)=\frac{l}{\pi a}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{k\pi at}{l}\sin\frac{k\pi x}{l}.$$

Bu qator yaqinlashmaydi, chunki $\frac{1}{k}$ qatori divergens.

Yuqoridagi masalani endi yagonalik shartiga tekshiramiz.

Teorema: Berilgan boshlang'ich va chegaraviy shartlar uchun to'lqin tenglamasining Dirixle masalasi yagona yechimga ega.

Isbot:

Faraz qilaylik, U_1 va U_2 ikkita yechim bo'lsin. $U = U_1 - U_2$ ayirma uchun

$U_{tt} - a^2U_{xx} = 0$ funksiya,

$U(0,t) = U(l,t) = 0$ chegaraviy va

$U_t(x,0) = 0, U(x,0) = 0$ boshlang'ich shartlar berilgan bo'lsin.

Energiya usuli orqali energiya funksionalini aniqlaymiz:

$$E(t) = \int_0^l \left(U^2{}_t + a^2U^2{}_x\right)$$

Energiyaning o'zgarishini hisoblaymiz:

$$\frac{dE}{dt} = \int_0^l \left(U_{tt}U_t + a^2U_xU_{xt}\right)dx = a^2\int_0^l \frac{\partial}{\partial x}\left(U_tU_x\right)dx = a^2\left[U_tU_x\right]_0^l = 0$$

Demak, $E(t) = const = E(0) = 0$, $E(t) = 0 \Rightarrow U_t = 0$ va $U_x = 0 \Rightarrow U = const$ ekanligi kelib chiqadi. Yuqoridagi masalaning yagona yechimi mavjud. Ammo bu masala turg'unlik shartini bajarmaydi.

Xulosa

Ushbu ishda to'lqin tenglamasi uchun Dirixle masalasining korrektivlik shartlari tahlil qilindi va bu shartlarning buzilishi natijasida masalaning nokorrekt holatga o'tishi ko'rsatildi. Tadqiqot natijalariga ko'ra, boshlang'ich va chegaraviy shartlarning to'liq bajarilishi yechimning mavjudligi, yagonaligi va barqarorligini ta'minlaydi. Dirixle shartining buzilishi esa yechim sezuvchanligini oshirib, amaliy hisoblarda jiddiy xatolarga olib kelishi mumkin. Shuningdek, energiya usuli yordamida masalaning yagonaligi isbotlandi. Bu natijalar matematik modellashtirish va raqamli hisoblashlarda muhim ahamiyat kasb etadi. Ushbu maqola “Nokorrekt va teskari masalalar” fanidan mustaqil ta’lim sifatida yozildi.

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