SELF-ADJOINT OPERATORS IN KREIN SPACES

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Abstract. *In this paper, positive, negative, and neutral elements and subspaces are defined using the inner product. The Krein space is defined, and the spectral properties of $J-$ self-adjoint operators in this space are studied. The relationships between self-adjoint operators in Hilbert space and $J-$ self-adjoint operators in Krein space are explored.*

Key words: *Inner product, Hilbert space, Krein space, Fundamental symmetry, J-orthonormal basis, self-adjoint operators.*

Introduction. Krein spaces represent a rich and intriguing generalization of Hilbert spaces in functional analysis. Unlike Hilbert spaces, which are characterized by positive-definite inner products, Krein spaces allow for indefinite inner products, meaning that the “length” (or norm) of a vector can be positive, negative, or zero.

This feature makes Krein spaces a powerful framework for addressing mathematical and physical problems that cannot be adequately handled within the confines of traditional Hilbert space theory.

A **Krein space** is essentially a vector space equipped with an indefinite inner product $(K, [\cdot,\cdot])$, which can be decomposed orthogonally into the direct sum of two Hilbert subspaces: $K = K_+ \oplus K_-$

Here, both $(K_+, [\cdot,\cdot])$ and $(K_-, -[\cdot,\cdot])$ are Hilbert spaces with positive-definite inner products, though the decomposition itself is not unique. Importantly, while the inner products may vary across decompositions, the resulting topologies (in terms of convergence, continuity, and other properties) remain equivalent.

Example 1. Let us consider the space $\mathbb{C}^2$, consisting of ordered pairs of complex numbers. In this space, we define an indefinite inner product as follows: $[x,y] = x_1\overline{y_1} - x_2\overline{y_2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

This space can be decomposed as: $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$,

where the first copy of $\mathbb{C}$ represents the subspace K_+, and the second copy represents the subspace K_-.

The subspace K_+ consists of elements of the form $(x_1, 0)$. In this subspace, the indefinite inner product becomes: $[x,y] = x_1\overline{y_1}$, which is the standard (positive-definite) inner product on complex numbers. Hence, $(K_+, [\cdot,\cdot])$ is a Hilbert space.

The subspace K_- consists of elements of the form $(0, x_2)$. In this subspace, the inner product is given by: $[x,y] = -x_2\overline{y_2}$. [1-2].

If we multiply this by a negative sign: $-[x,y] = x_2\overline{y_2}$ we recover the standard inner product on $\mathbb{C}$, indicating that $(K_-,\ -[\cdot,\cdot])$ is also a Hilbert space [3].

Therefore, the space $(\mathbb{C}^2, [\cdot,\cdot])$ admits a decomposition of the form:

$$\mathbb{C}^2 = K_+ \oplus K_-$$

and satisfies the conditions of a Krein space.

Example 2. Consider the space $L^2(\mathbb{R})$, consisting of square-integrable functions. With the standard inner product $[f,g] = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$ this space forms a Hilbert space [4-10].

Now, let us define an indefinite inner product on this space using the signum function $\operatorname{sgn}(t)$: $[f,g] = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\operatorname{sgn}(t)dt$

This inner product is indefinite, since the signum function $\operatorname{sgn}(t)$ can take both positive and negative values.

To make $L^2(\mathbb{R})$ into a Krein space, we decompose it into two as follows:

K_+: the subspace of functions that vanish for negative values of t, i.e., $f(t) = 0$ for $t < 0$;

K_-: the subspace of functions that vanish for positive values of t, i.e., $f(t) = 0$ for $t > 0$.

Thus, the space can be written as: $L^2(\mathbb{R}) = K_+ \oplus K_-$.

$(K_+, [\cdot,\cdot])$ is a Hilbert space: since $\mathrm{sgn}(t)$ is positive on K_+, the inner product reduces to the standard positive-definite form [11-18].

$(K_-, -[\cdot,\cdot])$ **is also a Hilbert space: on** K_-, where $\mathrm{sgn}(t)$ is negative, applying a minus sign to the inner product restores positive-definiteness.

Therefore, with this decomposition, $(L^2(\mathbb{R}), [\cdot,\cdot])$ becomes a Krein space.

Fundamental Symmetry. Let K be a Krein space with the orthogonal decomposition: $\boldsymbol{K} = \boldsymbol{K}_+ \oplus \boldsymbol{K}_-$.

Define the operator $J: K \to K$ by:

$$J(x_+ + x_-) = x_+ - x_-$$

where $x_+ \in K_+$ and $x_- \in K_-$. This operator is called the fundamental symmetry operator.

The operator J has the following properties [15-25]:

1. **Idempotency**: $J^2 = I$ where I is the identity operator.
2. **Self-adjointness**: $J^* = J$ meaning that J is self-adjoint with respect to the indefinite inner product.

Introducing a Hilbert Space Structure in a Krein Space: he fundamental symmetry operator $\boldsymbol{J}$ allows us to define a positive-definite inner product in the Krein space $\boldsymbol{K}$. This inner product is given by:

$$(x, y) = [Jx, y], \text{ for all } x, y \in K$$

This definition yields a Hilbert space structure, as it satisfies the following properties:

Positive definiteness: $(x, x) = [Jx, x] \geq 0$, and $(x, x) = 0$ if and only if $x = 0$.

Linearity and symmetry: The inner product is linear in the first argument and conjugate symmetric: $(x, y) = \overline{(y, x)}$.

Example 3. In the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$, the indefinite inner product is defined as: $[f, g] = \int_{-\infty}^{\infty} f(t)g(t)\operatorname{sgn}(t)dt$.

The corresponding fundamental symmetry operator $J: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined for all $f \in L^2(\mathbb{R})$ as:

$$(Jf)(t) = \operatorname{sgn}(t)f(t)$$

This operator acts by multiplying each function by the signum function, effectively flipping the sign of the function on the negative half-line.

The operator J has the following properties:

1. $J^2 = I$ **(Idempotency)**

Proof. Apply the operator J twice:

$$(J^2f)(t) = J(Jf)(t) = \operatorname{sgn}(t)(Jf)(t) = \operatorname{sgn}(t)\operatorname{sgn}(t)f(t)$$

Since $\operatorname{sgn}(t)^2 = 1$, we have: $J^2f = f$.

Thus, $J^2 = I$, where I is the identity operator.

2. $J^* = J$ **(Self-adjointness)**

Proof. Let us verify that J is self-adjoint with respect to the indefinite inner product, i.e., for all $f, g \in L^2(\mathbb{R})$, $[Jf, g] = [f, Jg]$.

Given that $Jf(t) = \operatorname{sgn}(t)f(t)$ we compute:

$$[Jf, g] = \int_{-\infty}^{\infty} (Jf)(t)\overline{g(t)}\operatorname{sgn}(t)dt = \int_{-\infty}^{\infty} \operatorname{sgn}(t)f(t)\overline{g(t)}\operatorname{sgn}(t)dt$$

$$= \int_{-\infty}^{\infty} f(t)\,\overline{\operatorname{sgn}(t)g(t)}\operatorname{sgn}(t)dt = \int_{-\infty}^{\infty} f(t)\overline{(Jg)(t)}sgn(t)dt = [f, Jg].$$

Therefore, J **is self-adjoint:** $J = J^*$.

The fundamental symmetry operator also allows us to define a Hilbert space inner product as:

$$(f,g) = [Jf,g] = \int_{-\infty}^{\infty} \text{sgn}(t) f(t)\overline{g(t)}\text{sgn}(t)dt$$

Since $\text{sgn}(t)^2 = 1$, this simplifies to the standard inner product on $L^2(\mathbb{R})$: $(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$

Thus, this inner product induces the strong topology of a Hilbert space on the Krein space $L^2(\mathbb{R})$.

The operator J **is called the fundamental symmetry operator, as it transforms the indefinite inner product in a Krein space into a positive-definite one. Using this operator, one can define a Hilbert space structure on the Krein space. The ability to reconstruct a Hilbert space topology from an indefinite inner product is of significant practical importance in physics and quantum mechanics [8-22].**

One of the key concepts in Krein spaces is the notion of a J-orthonormal basis.

$\boldsymbol{J}$ – Orthonormal bases**.** Let $(K,[\cdot,\cdot])$ be a Krein space with a fundamental symmetry operator $\boldsymbol{J}$. A sequence $\{\boldsymbol{e_n}\} \subset \boldsymbol{K}$ is called a $\boldsymbol{J}$-orthonormal basis if it satisfies the following conditions:

1. J – **orthonormality:** $[e_m, e_n] = f(x) = \begin{cases} 0, & m \neq n \\ \pm 1, & m = n \end{cases}$

2. **Completeness:** Every vector $x \in K$ can be represented in the form: $x = \sum_{n=1}^{\infty} J e_n$,

meaning that any vector in the space can be expressed using the J-orthonormal basis.

A J-orthonormal basis in a Krein space functions similarly to an orthonormal basis in a Hilbert space. Such bases:

- **Facilitate convenient representations of vectors and operators;**
- **Serve as a fundamental tool for spectral analysis and operator representation;**

- Exist in every Krein space—every Krein space possesses at least one J-orthonormal basis.

This type of basis plays a crucial role in analyzing elements within the Krein space and studying the spectral properties of operators.

Classification of Vectors and Subspaces: The presence of an indefinite inner product in Krein spaces leads to a classification of vectors and subspaces into the following categories:

Type of Vector or Subspace	Condition	Property
Positive vector	$[x,x] > 0$	Has a positive “length”
Negative vector	$[x,x] < 0$	Has a negative “length”
Neutral vector	$[x,x] = 0$	“Self-orthogonal”
Positive subspace	$\forall x \neq 0, [x,x] > 0$	All nonzero vectors have positive length
Negative subspace	$\forall x \neq 0, [x,x] < 0$	All nonzero vectors have negative length
Neutral subspace	$\forall x, [x,x] = 0$	All vectors are self-orthogonal

This classification is one of the fundamental distinctions between Krein spaces and Hilbert spaces, introducing the concepts of positive and negative vectors. Neutral vectors and subspaces do not appear in standard Hilbert spaces but play an important role in the theory of Krein spaces.

Definition 1. Let, $H = H_1 \oplus H_2$ be a Hilbert space, and let the operator

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1}$$

be defined on it.

If A is a densely defined linear operator on H such that the operator JA is self-adjoint in H, then A is called J-self-adjoint.

If JA is symmetric in the Hilbert space H, then A is called J-symmetric.

It is known that every J-symmetric operator is also J-self-adjoint.

If we define a new indefinite inner product on H by: $[\bullet,\bullet] = (J\ \bullet,\bullet)$ then a bounded linear operator A is J-self-adjoint if and only if:

$$[Ax, y] = [x, Ay], \qquad \text{for all } x, y \in H.$$

The Hilbert space H, equipped with the indefinite inner product $[\bullet.\bullet]$, becomes a Krein space, and every J-self-adjoint operator is self-adjoint with respect to this indefinite inner product.

Let A be a block operator matrix of the form: $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$.

Then, the operator A is self-adjoint if and only if the following conditions hold:

$$A_{11}^* = A_{11}, A_{22}^* = A_{22}, A_{21} = A_{12}^*$$

The operator A is J-self-adjoint if and only if:

$$A_{11}^* = A_{11}, A_{22}^* = A_{22}, A_{21} = -A_{12}^*.$$

Theorem 1. Assume that either $\dim H_1 \geq 2$ or $\dim H_2 \geq 2$. If the numerical range $W^2(A) \subset \mathbb{R}$ then the following hold: $A_{11}^* = A_{11}, A_{22}^* = A_{22}$, and either A has a block triangular form (i.e., $A_{12} = 0$ ·or $A_{21} = 0$), or there exists a real number $\gamma \in R, \gamma \neq 0$, such that: $A = \begin{pmatrix} A_{11} & A_{12} \\ \gamma A_{21} & A_{22} \end{pmatrix}$.

In the latter case, the operator A is similar to the block operator matrix:

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ \operatorname{sgn}(\gamma)\tilde{A}_{12} & A_{22} \end{pmatrix}, \qquad \tilde{A} = \sqrt{|\gamma|}A_{12}$$

If $\operatorname{sgn}(\gamma) = 1$, then $\tilde{A}$ is a J-self-adjoint operator.

Lemma 1. Let $A_{ij}: H_j \to H_i$, for $i, j = 1,2, i \neq j$, be densely defined closed operators such that

$$(A_{12}f_2, f_1)(A_{21}f_1, f_2) \in \mathbb{R}, \quad \forall f_j \in D(A_{ij}). \tag{2}$$

Then one of the following holds: $A_{12} = 0, A_{21} = 0$ or $A_{21} \subset \gamma A_{12}^*$, for some $\gamma \in \mathbb{R}$ (and $A_{21} = \gamma A_{12}^*$ if both A_{12} and A_{21} are bounded)

Proof. Suppose that $(A_{12}f_2, f_1) \neq 0$ for some $f_j \in D(A_{ij}), i.j = 1.2, i \neq j$. Then, from condition (2), we obtain [5-20]:

$$\frac{(A_{21}f_1, f_2)}{(f_1, A_{12}f_2)} = \frac{(A_{12}f_2, f_1)(A_{21}f_1, f_2)}{(A_{12}f_2, f_1)(f_1, A_{12}f_2)} \in R \quad (3)$$

Assume $A_{12} \neq 0$. Since A_{21} is densely defined, there exist elements $x_0 \in D(A_{21})$ and $y_0 \in D(A_{12})$ such that $(x_0, A_{12}y_0) \neq 0$. For arbitrary $u \in D(A_{21}), \gamma \in D(A_{12})$ define the function:

$$f_{u,\gamma}(z) = \frac{(A_{21}(x_0 + zu), y + \bar{z}\gamma)}{(x_0 + zu, A_{12}(y_0 + \bar{z}\gamma))} =$$

$$= \frac{(A_{21}x_0, y_0) + z\big((A_{21}x_0, \gamma) + (A_{21}u, y_0)\big) + z^2(A_{21}u, \gamma)}{(x_0, A_{12}y_0) + z\big((x_0, A_{12}\gamma) + (u, A_{12}y_0)\big) + z^2(u, A_{12}\gamma)}; z \in C$$

Since $(x_0, A_{12}y_0) \neq 0$, the denominator is not identically zero. Thus, $f_{u,\gamma}(\cdot)$ is a rational function on ℂ with at most two poles.

$$f_{u,\gamma}(z) = f_{u,\gamma}(0) = (A_2, x_0, y_0)/(x_0, A_{12}y_0) = \gamma \in R \text{ or}$$

$$(A_{21}x_0, y_0) + z\big((A_{21}x_0, v) + (A_{21}u, y_0)\big) + z^2(A_{21}u, v) =$$

$$= \gamma\left((x_0, A_{12}y_0) + z\big((x_0, A_{12}y_0) + (u, A_{12}v)\big) + z^2(u, A_{12}v)\right), z \in C\backslash\{\zeta_1, \zeta_2\}$$

By comparing the coefficients on both sides, for all $u \in D(A_{21}), v \in D(A_{12})$, we obtain: $(A_2, u, v) = \gamma(u, A_{12}v)$.

This implies $\gamma A_{12} \subset A_{21}^*$ or, equivalently, $A_{21} \subset \gamma A_{12}^*$. This completes the proof of Lemma 1.

Conclusion: Krein spaces provide a powerful theoretical framework for analyzing problems involving indefinite inner products. They serve as a generalization of Hilbert spaces and enable the study of phenomena that do not arise in conventional Hilbert space theory. The indefinite inner product introduces new classes of vectors, subspaces, and operators, giving rise to a distinct and intricate spectral theory [10-25].

The significance of Krein spaces extends beyond theoretical mathematics. They play an important role in various applied fields such as quantum mechanics, signal processing, and control theory. The ability to work with indefinite metrics makes Krein spaces a vital tool for analyzing complex systems and dynamic processes.

Although substantial research has already been conducted on Krein spaces, many open problems and research directions remain. These include the study of invariant subspaces, the development of numerical methods for Krein space operators, and the exploration of new applications across scientific disciplines.

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