BA'ZI AJOYIB TENGSIZLIKLAR VA ULARNI ISBOTLASH

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Annotatsiya: Maqolada musbat hamda ma'lum bir berilgan shartlarni qanoatlantiradigan sonlar uchun tengsizliklar hamda ularning isboti berilgan. Isbotlash maqsadida Koshi hamda Bernulli tengsizliklaridan foydalanilgan.

Kalit so'zlar: Koshi tengsizligi, musbat son, Chebishev tengsizligi, qavariq funksiya, Bernulli tengsizligi

Annotation: The article gives inequalities as well as proofs for numbers satisfying positive and certain given conditions. For proof purposes, Koshi as well as Bernoulli inequalities have been used.

Key words: Koshi inequality, positive number, Chebyshev inequality, convex function, Bernoulli inequality.

Musbat haqiqiy sonlar ustida berilgan shartlar asosida berilgan tengsizliklarning o'rinli ekanligini isbotlashning turlicha usullari mavjud ular ulardan eng ko'p qo'llaniladigan sonlarning o'rta arifmetigi hamda o'rta geometrigi haqidagi Koshi tengsizligidir. Bundan tashqari tengsizlikni kattalashtirib baholash yoki Chebishev tengsizligidan foydalanish kabi usullar ham mavjud. Bularning qaysi biridan foydalanish mutlaqo o'zimizga bog'liq. Quyida bir nechta tengsizliklar va ulaning isbotini ko'rib chiqamiz:

1-misol. Aytaylik, $x_1, x_2, \dots x_n$ haqiqiy musbat sonlar bo'lib,

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+\dots+\frac{1}{x_n+1998}=\frac{1}{x_1+1998}$$

tenglikni qanoatlantirsin,

$$\frac{\sqrt[n]{x_1x_2\dots x_n}}{n-1}\geq 1998$$

tengsizlikni isbotlang.

Yechilishi. $y_i=\frac{1}{x_1+1998}$ bo'lsin. U holda $y_1+\dots+y_n=\frac{1}{1998}$ va $x_i=\frac{1}{y_i}-1998$ bo'ladi. Quyidagi tenglik o'rinli:

$$\prod_{i=1}^{n} x_i=\prod_{i=1}^{n}\left(\frac{1}{y_i}-1998\right)=e^{\sum_{i=1}^{n} ln\left(\frac{1}{y_i}-1998\right)}$$

Demak, x_i larning ko'paytmasini mimimallashtirish, $ln(\frac{1}{y_i}-1998)$ lar yig'indisini minimallashtirishga ekvivalent. Bundan

$$\frac{d}{dy}(ln(\frac{1}{y}-1998)) = \frac{1}{(\frac{1}{y}-1998)^2}\cdot\frac{-1}{y^2} = \frac{-1}{y-1998y^2}$$

$$\frac{d^2}{dy^2}(ln(\frac{1}{y}-1998)) = \frac{1-3996y}{(y-1998y^2)^2}$$

tenglik o'rinli ekan.

Demak, $ln(\frac{1}{y}-1998)$ funksiya $\left[0;\frac{1}{3996}\right]$ da qavariq ekan. Agar barcha i lar uchun $0 < y_i < \frac{1}{3996}$ bo'lsa biz Jensen almashtirishidan foydalanishimiz mumkin edi. Demak, $y_i + y_j \le \frac{1}{1998}$ shartni qanoatlantiruvchi barcha i, j lar uchun tasodifiy olingan ixtiyoriy $\mathrm{a+b} \le \frac{1}{1998}$ uchun amal qiladigan quyidagi tengsizlikni ko'rib chiqamiz:

$$(\frac{1}{\mathrm{a}}-1998)(\frac{1}{\mathrm{b}}-1998) \ge (\frac{2}{\mathrm{a+b}}-1998)^2$$

$$\leftrightarrow \frac{1}{\mathrm{ab}} - 1998(\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{b}}) \ge \frac{4}{(\mathrm{a+b})^2} - \frac{4\cdot 1998}{(\mathrm{a+b})}$$

$$\leftrightarrow (\mathrm{a+b})^2 - 1998(\mathrm{a+b})^3 \ge 4\mathrm{ab} - 4\mathrm{ab}(\mathrm{a+b})\cdot 1998$$

$$\leftrightarrow (\mathrm{a-b})^2 \ge 1998(\mathrm{a+b})(\mathrm{a-b})^2$$

Shu bilan birga qaralayotgan yig'indini kamaytirish uchun har qanday ikkita y_i va y_j larning o'rta qiymatini mos qo'yish mumkin. Shuning uchun $y_i \in \left(0;\frac{1}{3996}\right]$ deb faraz qilamiz. Jensen tengsizligidan barcha i lar uchun $y_i = \frac{1}{1998n}$ bo'lganda yoki barcha i lar uchun $x_i = 1998(n-1)$ bo'lganda minimum qiymatga erishadi. Bundan esa tengsizlik osongina kelib chiqishi ko'rinib turibdi.

2-misol. $x_1x_2...x_n = 1$tenglik bilan aniqlangan barcha musbat haqiqiy $x_1, x_2, ... x_n$ lar uchun

$$\frac{1}{n-1+x_1} + ... + \frac{1}{n-1+x_n} \le 1$$

o'rinli bo'lishini ko'rsating.

Yechilishi: Dastlab quyidagi lemmani isbotlaymiz: Yig'indining maximal qiymati x_i $\mathrm{n}-1$ ga teng bo'lganda hosil bo'ladi. Ixtiyoriy nomanfiy k o'zgarmas son uchun $\mathrm{f(y)} = \frac{1}{\mathrm{k+e^y}}$ ni ko'rib chiqaylik. Bizda $f'(y) = \frac{e^{-y}}{(k+e^y)^2}$ va $f''(y) = \frac{e^y(e^y-k)}{(k+e^y)^3}$ o'rinli ekanligi bor. $f''(y) \ge 0 \leftrightarrow e^y \ge k$ o'rinli. Demak, f(y) funksiya $(\ln(\mathrm{k});\infty)$ da qavariq bo'ladigan $\mathrm{y = lnk}$ qiymatida f(y) funksiya yagona egilish nuqtasiga ega. Endi

$y_1 + ... + y_n = 0$ va $\sum_{i=1}^{n}\frac{1}{n-1+x_i} = \sum_{i=1}^{n} f(y_i)$ tengliklarni qanoatlantiradigan $y_i = ln(x_i)$ ketma ketlikni kiritamiz. Bundan $\mathrm{k = n-1}$ va $k_0 = ln(n-1)$ deb yozish mumkin. Ba'zi musbat m lar uchun $y_1 \geq \ldots \geq y_m \geq k_0 \geq y_{m+1} \geq \ldots x_n$ o'rinli deylik. Bundan quyidagi kattalashtirish o'rinli:

$$f(y_1)+\ldots+f(y_m) \leq (m-1)f(k_0) + f(y_1+\ldots+y_m-(m-1)k_0)$$

Shuningdek, $(m-1)f(k_0) + f(y_{m+1})+\ldots+f(y_n) \leq (n-1)f(\frac{(m-1)k_0+y_{m+1}+\ldots+y_n)}{n-1})$ kattalashtirish ham o'rinli boshqa tomondan barcha y_i lar k_0 lardan kichik. Shu sababli biz y_i ni $n-1$ga tenglashtirish uchun kattalashtirishni qo'llaymiz, shu bilan birga ko'rib chiqilayotgan yig'indini oshiramiz. Demak, lemma isbotlandi.

Lemmani qo'llab $\frac{k}{k+x} + \frac{k}{k+\frac{1}{x^k}} \leq 1$ tengsizlikning to'g'riligini ko'rsatishimiz yetarli.

Maxrajlardan qutulamiz,

$$\left(k^2 + \frac{k}{x^k}\right) + (k+x) \leq k^2 + k\left(x + \frac{1}{x^k}\right) + x^{1-k} - xk + x + k \leq x^{1-k}$$

Ammo bu aniq edi. Bernulli tengsizligidan

$x^{1-k} = (1+(x-1)^{1-k} > 1 + (x-1)(1-k) = x + k - xk$ o'rinli. Bundagi tenglik faqat x = 1 yoki n = 2 da bajariladi.

3-misol. Barcha musbat haqiqiy a, b, c sonlari uchun

$$\frac{\sqrt{b+c}}{a} + \frac{\sqrt{a+c}}{b} + \frac{\sqrt{b+a}}{c} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}$$

tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi.

1-usul. Koshi tengsizligiga ko'ra, $\sqrt{(a+b)(c+a)} \geq a + \sqrt{bc}$. o'rinli bo'lishi aniq . Bundan quyidagini yozish mumkin:

$$\sum_{cyc} \frac{\sqrt{b+c}}{a} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}$$

$$\leftrightarrow \sum_{cyc} \frac{b+c}{a}\sqrt{(a+b)(c+a)} \geq 4(a+b+c)$$

Koshi tengsizligidan kelib chiqadigan natijani almashtiramiz va

$\sum_{cyc}(b+c)\frac{\sqrt{bc}}{a} \geq 2(a+b+c)$ tengsizlikni ko'rsatishimiz yetarli.

a > b > c ekanligidan $b + c \leq c + a \leq a + b$ va $\frac{\sqrt{bc}}{a} \leq \frac{\sqrt{ac}}{b} \leq \frac{\sqrt{ba}}{c}$ o'rinli. AM-GM va Chebishev tengsizligidan

2-usul. Aytaylik, $x = \sqrt{b+c}$, $y = \sqrt{a+c}$, $z = \sqrt{b+a}$ bo'lsin. x, y, z sonlar XYZ o'tkir burchakli uchburchakning tomonlari deb olsak, $x^2 + y^2 = a + b + 2c > a + b = z^2$ o'rinli. Yuqoridagi tengsizlik esa

$$\sum_{cyc} \frac{x}{y^2+z^2-x^2} \geq \frac{x^2+y^2+z^2}{xyz}$$

Tengsizlikka ekvivalentdir. $y^2 + z^2 - x^2 = 2yzcos(X)$ ifoda o'rinli ekanligidan bu ekvivalent ifodani kamaytiramiz:

$$\sum_{cyc} \frac{x^2}{cos(X)} \geq 2(x^2 + y^2 + z^2)$$

x ≥ y ≥ z ekanligidan $\frac{1}{cos(X)} \geq \frac{1}{cos(Y)} \geq \frac{1}{cos(Z)}$ o'rinli natijada chap tomonga Chebishev tengsizligini qo'llasak, kosinuslar o'zaro yig'indisining kamida 6 ekanligini isbotlash uchun kerakli miqdorni kamaytiradi, ya'ni AM-HM dan

$\frac{1}{cos(X)} + \frac{1}{cos(Y)} + \frac{1}{cos(Z)} \geq \frac{9}{cos(X)+cos(Y)+cos(Z)}$ o'rinli . Ammo geometriyada Uchburchak qoidalarini eslasak, $cos(X) + cos(Y) + cos(Z) = 1 + \frac{r}{R}$ va R ≥ 2r tengsizlik bajarilar edi, demak faraz qilgan holatimiz isbotlandi.

4-misol. Barcha musbat haqiqiy $x_1, x_2, \ldots x_n$ uchun

$\frac{{x_1}^3}{{x_1}^2+x_1x_2+{x_2}^2} + \frac{{x_2}^3}{{x_2}^2+x_2x_3+{x_3}^2} + \ldots + \frac{{x_n}^3}{{x_n}^2+x_nx_1+{x_1}^2} \geq \frac{x_1+\ldots+x_n}{3}$ tengsizlikni isbotlang.

Yechilishi. Bizga yaxshi ma'lumki,

$0 = (x_1\text{-}x_2) + (x_2 - x_3) + \ldots + (x_n - x_1) = \sum_{i=1}^{n} \frac{{x_i}^3 - {x_{i+1}}^3}{{x_i}^2+x_ix_{i+1}+{x_{i+1}}^2}$ tenglik o'rinli (bu yerda $x_{n+1} = x_1$). Shuning uchun

$\sum_{i=1}^{n} \frac{{x_i}^3}{{x_i}^2+x_ix_{i+1}+{x_{i+1}}^2} = \frac{1}{2}\sum_{i=1}^{n} \frac{{x_i}^3+{x_{i+1}}^3}{{x_i}^2+x_ix_{i+1}+{x_{i+1}}^2}$ tenglik bajariladi.

Ammo $a^3 + b^3 \geq \frac{1}{3}a^3 + \frac{2}{3}a^2b + \frac{2}{3}ab^2 + \frac{1}{3}b^3 = \frac{1}{3}(a+b)(a^2+ab+b^2)$ tenglikdan

$\frac{1}{2}\sum_{i=1}^{n} \frac{{x_i}^3+{x_{i+1}}^3}{{x_i}^2+x_ix_{i+1}+{x_{i+1}}^2} \geq \frac{1}{2}\sum_{i=1}^{n} \frac{x_i+x_{i+3}}{3} = \frac{1}{3}\sum_{i=1}^{n} x_i$ o'rinli ekanligini aytish mumkin.

5-misol. Aytaylik, a, b, c musbat haqiqiy sonlar bo'lib, a + b ≥ c;

b + c ≥ a; c + a ≥ b bo'lsin. U holda

$$2a^2(b+c) + 2b^2(a+c) + 2c^2(b+a) \geq a^3 + b^3 + c^3 + 9abc$$

tengsizlik bajarilishini ko'rsating.

Yechilishi. Tengsizlikni isbotlash uchun a = y + z, b = z + x, c = x + y almashtirishni bajaramiz. Chap tomondagi ifoda

$4x^3 + 4y^3 + 4z^3 + 10x^2(y+z) + 10y^2(z+x) + 10z^2(x+y) + 24xyz$

ko'rinishga, o'ng tomondagi ifoda esa

$2x^3 + 2y^3 + 2z^3 + 12x^2(y+z) + 12y^2(z+x) + 12z^2(x+y) + 18xyz$

ko'rinishga keladi. Bu ifoda Shur tengsizligi deb ataluvchi

$$x^3 + y^3 + z^3 + 3xyz \geq x^2(y+z) + y^2(z+x) + z^2(x+y)$$

tengsizlikka ekvivalentdir.

Tenglik x = y = z bo'lganda ya'ni (a, b, c) = (t, t, t) yoki x, y, z larni ikkitasi teng, uchinchisi nol bo'lganda ya'ni (a, b, c)϶{(2t, t, t), (t, 2t, t), (t, t, 2t)} bo'lganda bajariladi.

6-misol. a, b, c sonlari uchburchak tomonlari bo'lsa,

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}}+\frac{b}{\sqrt{2a^2+2c^2-b^2}}+\frac{c}{\sqrt{2b^2+2a^2-c^2}} \geq \sqrt{3}$$

Yechilishi. Dastlab $a = y + z, b = z + x, c = x + y$ almashtirish bajaramiz, bizga yaxshi ma'lumki x, y, z musbat sonlar. Demak,

$$\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}=\sum_{cyc}\frac{y+z}{\sqrt{4x^2+4xy+4xz+y^2+z^2-2yz}}$$

tenglik o'rinli.

$f(x) = \frac{1}{\sqrt{x}}$ qavariq funksiyani ko'rib chiqaylik. (Ko'rinib turibdiki, Jusen har doim tengsizliklardan radikallarni yo'q qilish uchun qulay almashtirishni taqdim qiladi.) $x + y + z = 1$ bo'lsin, u holda

$$\sum_{cyc}(y+z)f(4x^2+4xy+4xz+y^2+z^2-2yz) \geq$$

$$((y+z)+(z+x)+(x+y))f\left(\frac{\sum_{cyc}(y+z)(4x^2+4xy+4xz+y^2+z^2-2yz)}{(y+z)+(z+x)+(x+y)}\right)$$

$$=\frac{2\sqrt{2}}{\sqrt{\sum_{cyc}4x^2(y+z)+(4xy^2+4xyz)+(4xyz+4xz^2)+y^3+z^3-y^2z-yz^2}}$$

$$\sum_{cyc}4x^2(y+z)+(4xy^2+4xyz)+(4xyz+4xz^2)+y^3+z^3-y^2z-yz^2$$

$$=\sum_{cyc}2x^3+7x^2(y+z)+8xyz,$$

$$8(x+y+z)^3 \geq 3\sum_{cyc}2x^3+7x^2(y+z)+8xyz,$$

$$\leftrightarrow \sum_{sym}4x^3+24x^2y+8xyz \geq \sum_{sym}3x^3+21x^2y+12xyz \geq$$

$$\leftrightarrow 2x^3+2y^3+2z^3+3(x^2(y+z)+y^2(x+z)+z^2(y+x)) \geq 24xyz$$

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