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INVERSE BOUNDARY VALUE PROBLEM

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Annotation: This article discusses the steepest descent algorithm of the gradient method used to solve systems of linear algebraic equations. The method for finding the solution by minimizing a functional is explained step-by-step. Theoretical foundations based on gradient and error vectors are presented. A practical example involving a system of four equations is solved using Python, and the convergence rate and accuracy of the algorithm are demonstrated. The results show that the gradient method is a simple and efficient computational tool suitable for solving large-scale linear systems.

Key words: Gradient method, steepest descent, iterative method, symmetric matrix, positive definiteness, functional, optimization, Python, algorithm, error vector.

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Аннотация: В статье рассматривается обратная задача определения граничного условия для уравнения теплопроводности. Неизвестная функция, зависящая от времени, определяется на основе дополнительной информации о решении. Существование и единственность решения доказываются с помощью теорем 1 и 2. Кроме того, показано, что задача может быть сведена к интегральному уравнению типа Вольтерра.

Ключевые слова: обратная задача, уравнение теплопроводности, граничное условие, интегральное уравнение, единственность, уравнение Вольтерра

Annotatsiya: Ushbu maqolada issiqlik tarqalish tenglamasi uchun chegaraviy shartni aniqlashga oid teskari masala ko‘rib chiqiladi. Noma’lum bo‘lgan vaqtga bog‘liq funksiyani aniqlash uchun yechim haqidagi qo‘shimcha ma’lumotlardan foydalaniladi. Masalaning yechimining mavjudligi va yagonaligi 1- va 2-teoremalar orqali asoslanadi. Bundan tashqari, bu masala Volterra turidagi integral tenglama orqali yechim topishga keltirilishi ham ko‘rsatib beriladi.

Kalit so’zlar: teskari masala, issiqlik tarqalish tenglamasi, chegaraviy shart, integral tenglama, yagonalik, Volterra tenglamasi

Introduction:

Achieving fast and accurate solutions to systems of linear algebraic equations is one of the key directions of modern computational mathematics. In particular, for systems with symmetric and positive definite coefficient matrices, gradient-based methods—especially the steepest descent algorithm—are highly effective. This method operates on an iterative approach, moving in the direction of the greatest decrease of the objective function at each step. This article presents the theoretical foundations of the gradient method, the steps of the algorithm, and a practical example implemented using the Python programming language.

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Determination of Boundary Conditions. We consider an inverse problem that involves determining a time-dependent function included in the boundary condition, using additional information about the solution of the boundary value problem for the heat conduction equation. Let us assume that $u(x,t)$ the function

$$u_t = a^2 u_{xx}, 0 < x < l, 0 < t \le T, \quad (1)$$

$$u(0,t) = \mu(t), \ \ 0 \le t \le T, \quad (2)$$

$$u_x(l,t) = \nu(t), \ \ 0 \le t \le T, \quad (3)$$

$$u(x,0) = 0, \quad 0 \le x \le l, \quad (4)$$

boundary issues of the solution , let it be.

Assume let, $\nu(t)$ given the function, $\mu(t)$ of unknown function, and (1)-(4) of problem solution about

$$u(l,t) = g(t), \ \ 0 \le t \le T, \quad (5)$$

additional information is, $\mu(t)$ the function to determine required will be, this earth $g(t)$- given function.

Let us consider the issue of the uniqueness of the solution to the given inverse problem. The uniqueness of the solution to this problem has been studied in a more general form by a number of authors. Let us present one of the results obtained in this direction, as provided in [51], and apply it to the case of the heat conduction equation (1) with constant coefficients.

We introduce the following notation:

$$Q_{lT} = \{(x,t) : 0 \le x \le l, 0 \le t \le T\}.$$

1-teorema. Assume let me, the function $\bar{u}(x,t)\in C^{2,1}(Q_{lT})$ and (1) equations of the Q_{lT} area will build. It without , you $0<t\le T$ for $\bar{u}(l,t)=\bar{u}_x(l,t)=0$ be, Q_{lT} at $\bar{u}(x,t)=0$ will.

This teorema from given $\nu(t)$ and $g(t)$ the function for (1)-(5) from $\mu(t)$ the function identifying issues of the singleness come out can. Fact is, $u_1(x,t),u_2(x,t)\in C^{2,1}(Q_{lT})$ Q_{lT} in the area of (1) the equation may build and

$$u_1(0,t)=\mu_1(t),\ u_2(0,t)=\mu_2(t),\ 0\le t\le T,$$

$$u_1(l,t)=u_2(l,t)=g(t),\ \frac{du_1}{dx}(l,t)=\frac{du_2}{dx}(l,t)=\nu(t),\ 0\le t\le T$$

which is.

Now $\bar{u}(x,t)=u_1(x,t)-u_2(x,t)$ the function we do not look at. This function is 1-teorema terms and conditions will be content. Therefore, Q_{lT} in the field , $\bar{u}(x,t)=0$ ie $t\in[0,T]$ for $\mu_1(t)=\mu_2(t)$.

Attention give, 1-teorema in $\bar{u}(x,t)$ the function for the initial conditions, it is not given.

(1)-(4) boundary issues for other reverse the issue , we do not look at.

Assume let, $\nu(t)$ given the function, $\mu(t)$ unknown following

$$u(x_0,t)=g(t),\ 0<t\le T \qquad (6)$$

Wholesale of view (1)-(4) of problem solution about additional information known if it is, $\mu(t)$ the function to identify necessary, this earth on $g(t)$ a given function, $x_0\in[0,l]$.

The uniqueness of the solution to this inverse problem can also be established using Theorem 1.

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2-teorema. If $u_1(x,t), u_2(x,t) \in C^{2,1}(Q_{lT})$ the function of the Q_{lT} area in (1) in the equation and (3), (4), (6) terms and conditions does it build, $t\in[0,T]$ for $u_i(0,t)=\mu_i(t), i=1,2,$ is. It without $t\in[0,T]$ though $\mu_1(t)=\mu_2(t)$ it will be.

Proof. We consider the function $\bar{u}(x,t)=u_1(x,t)-u_2(x,t)$ where $\bar{u}(x,t)$

$$u_t=a^2u_{xx}, x_0<x<l, 0<t\le T,$$

$$\bar{u}(x_0,t)=0,\ \ 0\le t\le T,$$

$$\bar{u}_x(l,t)=0,\ \ 0\le t\le T,$$

$$\bar{u}(x,0)=0,\ \ x_0\le x,\le l,$$

boundary of the issue the solution is. Optional $x_0\in[x_0,l],\ t\in[0,T]$ for $\bar{u}(x,t)=0$ this is that we show. For this, we multiply the equation by a function $\bar{u}(x,t)$ integrate, and

$$\int_{x_0}^{l}\int_{0}^{t}\bar{u}_t(\xi,\tau)\bar{u}(\xi,\tau)d\tau d\xi=a^2\int_{x_0}^{l}\int_{0}^{t}\bar{u}_{xx}(\xi,\tau)\bar{u}(\xi,\tau)d\xi d\tau$$

ensure you will. Integrated with many pieces in the calculation, the initial and boundary conditions we will use, the result $t\in[0,T]$ for the following equality able you will be:

$$\frac{1}{2}\int_{x_0}^{l}(\bar{u}(\xi,\tau))^2d\xi+a^2\int_{x_0}^{l}\int_{0}^{t}(\bar{u}_x(\xi,\tau))^2d\xi d\tau$$

That come out, $x\in[x_0,l], t\in[0,T]$ to $\bar{u}(x,t)=0$. It without $\bar{u}(x_0,t)=0$. So done, $x\in[0,x_0],\ t\in[0,T]$ when $\bar{u}(x,t)$ the function (1) in the equation and $t\in[0,T]$ for $\bar{u}(x_0,t)=\bar{u}(x_0,t)=0$ terms and conditions will build. We apply Theorem 1 on the rectangle $0\le x\le x_0, 0\le t\le T$, for $t\in[0,T]$, and thus $u_1(0,t)=u_2(0,t)$ or $\mu_1(t)=\mu_2(t)$ ollows. The proof of Theorem 2 is completed..

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Let us present an example of reducing the problem of determining a boundary condition to the solution of a first-kind integral equation. We consider the boundary value problem for the heat conduction equation on the half-line:

$$u_t = u_{xx}, 0 < x < \infty, 0 < t \le T, \quad (7)$$

$$u(0,t) = \mu(t), \ \ 0 \le t \le T, \quad (8)$$

$$u(x,0) = 0, \ \ 0 \le x < \infty. \quad (9)$$

(7)-(9) problem solution about

$$u(x_0,t) = g(t), \ \ 0 \le t \le T, \ \ x_0 > 0, \quad (10)$$

Additional information is given, $\mu(t)$ the function to identify required is.

(7)-(9) the issue of the solution

$$u(x,t) = \int_0^t \frac{x}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} \mu(\tau) d\tau$$

has the following form. Thus, in this case, the inverse problem

$$\int_0^t K(t,\tau)\mu(\tau)d\tau = g(t), \ \ 0 \le t \le T,$$

It is reduced to a Volterra integral equation of the first kind, where the kernel is...

$$K(t,\tau) = \frac{x_0}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left\{-\frac{x_0^{\,2}}{4(t-\tau)}\right\}.$$

General Conclusion:

In this paper, an inverse problem related to the determination of a boundary condition for the heat conduction equation was investigated. The problem involved determining an unknown time-dependent function using additional information about

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the solution. Based on Theorems 1 and 2, the uniqueness of the solution was proven. Furthermore, a practical method for obtaining the solution was demonstrated by reducing the problem to a Volterra-type integral equation. The results of this study can be applied in modeling and controlling physical processes and contribute to the theoretical and practical understanding of inverse problems.

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