

TALABALAR OLIMPIADASINING SARALANGAN MASALALARI VA
ULARNING YECHIMLARI.

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Annotatsiya. Ushbu maqolada talabalar olimpiadasining sara masalalari va ularning yechimlari keltirilgan. Maqolada matematik analiz, geometriya, algebra kurslariga oid masalalar jamlangan. Maqolada keltirilgan masalalar va ularning yechimlarini o'rganish akademik-litsey, kollej, institut, universitet talabalari va matematika qiziquvchi keng jamoatchilik uchun maanfatli bo'ladi degan umiddamiz.

Kirish.

1. Agar $a, b, c \in N$ va $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$ bo'lsa, $a \cdot b \cdot c$ biror natural sonning kubi ekanligini ko'rsatilsin.

Isbot. O'rta arifmetik va o'rta geometrik orasidagi tengsizlikka ko'ra:

$$3 = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 3 \rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3 \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} \leftrightarrow \frac{a}{b} = \frac{b}{c} = \frac{c}{a} \rightarrow$$

$$a^2 = bc, \quad b^2 = ac, \quad c^2 = ab$$

$$a^3 = abc, \quad b^3 = abc, \quad c^3 = abc$$

2. Quyidagi tenglamaning cheksiz ko'p natural yechimi topilsin.

$$x^{2000} + y^{2000} + z^{2000} = u^{2001}$$

$$x = y = z = u = 3$$

Yechim. Endi $x = y = z = 3^k$, $u = 3^n$ desak

$$(3)^{2000k} + (3)^{2000k} + (3)^{2000k} = (3)^{2001n} \rightarrow$$

$$3 \cdot 3^{2000k} = 3^{2001n} \rightarrow 2000k + 1 = 2001n \rightarrow 2000k = 2001n - 1$$

bu tenglikning chap tomoni juft, demak, $n = 2m + 1$ (ya'ni n-toq) bo'lishi kerak.

Bundan kelib chiqadiki,

$$2000k = 2001(2m + 1) - 1 \rightarrow 2000k = 4002m + 2000 \rightarrow 1000k$$

$$= 2001m + 1000 \rightarrow$$

$$k = 2m + 1 + \frac{m}{1000} (*)$$

Bu yerdan ko‘rinadiki (*) tenglama natural sonlar to‘plamda cheksiz ko‘p yechimga ega, ya’ni m ning 1000 ga karrali har bir qiymatida k -natural bo‘ladi. Demak, m ning 1000 karrali har bir qiymatida n topiladi va k topiladi. Bu esa tenglama cheksiz ko‘p yechimga ega ekanligini ko‘rsatadi.

3. Perimetri 6 cm bo‘lgan ixtiyoriy uchburchakda, uning yuzasi $\sqrt{3}$ dan katta bo‘lmasligini ko‘rsating.

Isbot.

I-usul. x, y, z – uchburchak tomonlari bo‘lsin,

$$6 = x + y + z \geq \sqrt[3]{xyz}$$

$$\sqrt[3]{xyz} \leq 2$$

$$xyz \leq 8$$

$$s = \frac{1}{2}xy \sin \gamma = \frac{1}{2}xz \sin \beta = \frac{1}{2}yz \sin \alpha$$

$$s^3 = \frac{1}{8}(xyz)^2 \sin \alpha \sin \beta \sin \gamma \leq \frac{1}{8}8^2 \sin \alpha \sin \beta \sin \gamma = 8 \sin \alpha \sin \beta \sin \gamma$$

Endi, $\alpha + \beta + \gamma = 180^\circ$ shartda $\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8}$ ekanligini ko‘rsatamiz.

$$\begin{aligned} \sin \alpha \sin \beta \sin \gamma &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \sin \gamma \\ &= \frac{1}{2} [\cos(\alpha - \beta) + 1 - 1 + \cos \gamma] \sin \gamma \cos^2 \frac{\alpha - \beta}{2} \sin \gamma - \sin^2 \frac{\gamma}{2} \sin \gamma \\ &\leq \sin \gamma - \sin^2 \frac{\gamma}{2} \sin \gamma = \sin \gamma \left(1 - \sin^2 \frac{\gamma}{2}\right) = 2 \cos^3 \frac{\gamma}{2} \sin \frac{\gamma}{2} \\ &= 2 \left[\left(1 - \sin^2 \frac{\gamma}{2}\right)^{1/2} \right]^3 \sin \gamma \end{aligned}$$

Endi $f(x) = 2(1 - x^2)^{3/4}x$ funksiyaning eng katta qiymatini topamiz.

$f'(x) = \sqrt{1 - x^2}(2 - 8x^2)$, $f'(x) = 0$ ekanligidan, tenglamaning ildizlari

$x_{1,2} = \pm 1$ va $x_{3,4} = \pm \frac{1}{2}$ va $\sin \frac{\gamma}{2} = x \geq 0$ bo‘lganligi uchun $f(1) = 0$, $f\left(\frac{1}{2}\right) =$

$\frac{3\sqrt{3}}{8}$, demak $\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8}$ bundan esa

$$s^3 = (\sqrt{3})^3 \rightarrow s \leq \sqrt{3}$$

II-usul. Perimetri o'zgarmas bo'lgan uchburchaklar ichida yuzasi eng katta bo'lgani teng tomonli uchburchak ekanligini ko'rsatamiz. Bu esa o'z navbatida quyidagi algebraik tasdiqqa suyanadi.

Tasdiq. Uchta manfiymas sonlar x, y, z larning yig'indisi o'zgarmas bo'lsa, ularning ko'paytmasi eng katta bo'ladi, agar ular teng bo'lsa.

Isbot. $x + y + z \geq 3\sqrt[3]{xyz}$, $x + y + z = p$ tenglik bajariladi agarda $x = y = z$ bo'lsa

Ikki manfiymas sonlar yig'indisi o'zgarmas bo'lsa ularning ko'paytmasi eng katta bo'lishi uchun ular teng bo'lishi zarur. $x + y = p$, $x = p - y$ $xy = (p - y)y$ kvadratik funksiya eng katta qiymatini $y = \frac{p}{2}$,

$xy = \frac{p^2}{4}$ eng katta bo'ladi

Endi shuni tasdiqqa qo'llaymiz. $x + y + z = p$

$$x + (y + z) = p, \quad x = y + z = \frac{p}{2}$$

$$(x + y) + z = p, \quad x + y = z = \frac{p}{2}$$

$$(x + z) + y = p, \quad x + z = y = \frac{p}{2}$$

$x = y = z$ tasdiq isbot qiladi.

Yuqoridagi tasdiqqa asosan masalani II-usul bilan yechamiz. Shartga asosan uchburchak tomonlari,

x, y, z bo'lsin.

$$p = x + y + z = 6, \quad s = \sqrt{\frac{p}{2} \left(\frac{p}{2} - x\right) \left(\frac{p}{2} - y\right) \left(\frac{p}{2} - z\right)} = \sqrt{3(3-x)(3-y)(3-z)}$$

$(3-x) + (3-y) + (3-z) = 9 - (x+y+z) = 3$ ekanligidan $(3-x)(3-y)(3-z)$ eng katta bo'lishi uchun $3-x = 3-y = 3-z$ ya'ni $x = y = z = 2$

4. Yuзи $\sqrt{3} \text{ cm}^2$ bo‘lgan ixtiyoriy uchburchak perimetri quyidan qaysi son bilan chegaralangan, yuqoridanchi?

Yechim. Quyidagi tasdiq o‘rinli.

Tasdiq. Bir nechta manfiy mas sonlar ko‘paytmasi o‘zgarimas bo‘lsa, ularning yig‘indisi eng kichik bo‘ladi. Agar bu sonlar teng bo‘lsa. Shunga asosan

$$S = \sqrt{p(p-x)(p-y)(p-z)} = \sqrt{3}$$

ya’ni, $p(p-x)(p-y)(p-z) = 3$

uchburchak tomonlari $p-x = p-y = p-z$ bo‘lganda, ya’ni $x = y = z$ holda

$$p-x + p-y + p-z = 3p - (x+y+z) = p$$

$$p = \frac{3x}{2} = \frac{3y}{2} = \frac{3z}{2} \text{ bo‘lganda, eng kichik bo‘ladi.}$$

$$p(p-x)^3 = 3 \text{ bundan } p\left(p - \frac{2}{3}p\right)^3 = p \cdot \frac{p^3}{27} = \frac{p^4}{27} = 3$$

$$p^4 = 81 \rightarrow p = 3$$

Perimetr $p = 6, 6 \leq p \leq \infty$

Tasdiq isboti o‘rta arifmetik va o‘rta geometrik haqidagi teoremlardan kelib chiqadi.

5. Ixtiyoriy $n \in N$ uchun $a_n = n^2 + n + 1$ formula bilan berilgan ketma-ketlikning ixtiyoriy ikkita ketma-ket hadi ko‘paytmasi, shu ketma-ketlikning qaysidir hadiga teng bo‘lishini ko‘rsating.

Isbot. $a_n = n^2 + n + 1, a_{n+1} = (n+1)^2 + (n+1) + 1$

$$a_n a_{n+1} = (n^2 + n + 1)(n^3 + 3n + 3) = n^4 + 4n^3 + 7n^2 + 6n + 3$$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1 + n^2 + 2n + 2$$

$$= (n+4)^4 + (n+1)^2 + 1 = a_{(n+1)}^2$$

6. $\triangle ABC$ ga AD – mediana

$$\frac{BE}{EC} = \frac{2}{1}, \angle ADC = \angle BAE \quad \angle BAC = ?$$

Yechim. $\angle ABC = \angle BAE$ va $\angle AED$, $\triangle AED$ va $\triangle BAE$ uchun umumiylikidan $\triangle AED \sim \triangle BAE$ o‘xshashlikka asosan

$$\frac{BE}{AE} = \frac{AE}{ED} \rightarrow AE^2 = BE \cdot ED = 2x \frac{x}{2} = \frac{x^2}{2} \rightarrow AE = x = CE$$

$CB = 3x$ desak, $BE = 2x$, $CE = x$, $BD = DC = \frac{3x}{2}$ ekanligidan $ED = EB -$

$$BD = 2x - \frac{3x}{2} = \frac{x}{2}$$

Xulosa qilib aytganda $\angle AEC$ – teng yonli. DB tomonda E_1 nuqta olamiz, u quyidagicha olinadi:

$$ED = DE_1, \frac{AE}{AD} = \frac{AB}{AD} = 2 \text{ o'xshashlikka ko'ra.}$$

Bundan tashqari $\frac{BE_1}{E_1D} = 2$, chunki $BE_1 = BD - ED = \frac{3x}{2} - \frac{x}{2} = x$ demak

$$\frac{AD}{AB} = \frac{E_1D}{BE_1} \text{ dan } AE_1 \text{ – bissektressa ekanligi kelib chiqadi. } ED = DE_1 \text{ ekanligidan}$$

esa $\angle EAD = \angle DAE_1 = \angle E_1AB = \beta$

$$AE = AE_1 = x = E_1B \text{ dan } \angle ABD = \beta, \angle ACE = \gamma \text{ desak } 2\gamma + 4\beta = 180^\circ \rightarrow \gamma = 90^\circ - 2\beta$$

sinuslar teoremasiga ko'ra

$$\frac{AE}{\sin \beta} = \frac{BE}{\sin 3\beta} \rightarrow \frac{\sin 3\beta}{\sin \beta} = 2$$

$$\frac{\sin \beta (4 \cos^2 \beta - 1)}{\sin \beta} = 2$$

$$\cos \beta = \pm \frac{\sqrt{3}}{2}, \beta = 30^\circ, \gamma = 30^\circ \rightarrow \angle ACE = 30^\circ.$$

7. $0 \leq a, b, c, d \leq 1$ bo'lsa $ab + bc + abcd - a - b - abc - bcd$ ifodaning maksimal qiymatini toping.

$$\begin{aligned} \text{Yechim. } ab + bc + abcd - a - b - abc - bcd &= b(a + c + acd) - b(1 + ac + cd) - a \\ &= b(a + c + acd - 1 - ac - cd) - a = b(a - 1 + c(1 - a) + cd(a - 1)) - a \\ &= b(a - 1)(1 - c + cd) - a \leq 0 \end{aligned}$$

Tengliklardan va $a - 1 \leq 0$, $-a \leq 0$ ekanligidan eng katta qiymat nolga teng ekan.

8. Perimetri 2 ga teng bo'lgan uchburchak tomonlari a, b, c bo'lsa, $a^2 + b^2 + c^2 + 2abc < 2$ ekanligini isbotlang.

Isbot. $a + b + c = 2 \rightarrow a = 2 - b - c$ ifodani $a^2 + b^2 + c^2 + 2abc$ ga qo'ysak,

$$\begin{aligned} & (2 - b - c)^2 + b^2 + c^2 + 2(2 - b - c) \\ &= 4 + b^2 + c^2 - 4b - 4c + 2bc + b^2 + c^2 + 4bc - 2b^2c - 2bc^2 \\ &= 2b^2 + 2c^2 - 2b^2c - 2bc^2 + 6bc - 4b - 4c + 4 \\ &= 2b^2 + 2c^2 + 4bc + 2bc - 2b^2c - 2bc^2 - 4(b + c - 1) \\ &= 2(b + c)^2 + 2bc(1 - c - b) - 4(b + c - 1) \\ &= 2(b + c)^2 + (b + c - 1)(-2bc - 4) < 2(b + c)^2 < 2 \end{aligned}$$

(Chunki $b + c > 1$ bu $a, b, c < 1$ dan kelib chiqadi.

9. a va b natural sonlar bo'lsin. ab – juft bo'lganda, shunday c va d natural sonlar topilib ular uchun $a^2 + b^2 + c^2 = d^2$ tenglik bajarilishini, agarda ab – juft bo'lmaganda bo'ldanda bunday c va d larning topilmasligini isbotlang.

Isbot. ab – juft,

10. $1, x_1, x_2, \dots, x_{n-1}$ sonlar $x^n = 1$ tenglamaning ildizlari bo'lsa, quyidagini isbotlang:

$$(1 - x_1)(1 - x_2) \dots (1 - x_{n-1}) = n$$

Isbot. Shartga ko'ra,

$$x^n - 1 = (x - 1)(x - x_1) \dots (x - x_{n-1})$$

Ammo,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$

Quyidagini bilgan holda

$$(x - x_1)(x - x_2) \dots (x - x_{n-1}) = x^{n-1} + x^{n-2} + \dots + x + 1$$

$x=1$ nuqtada quyidagiga ega bo'lamiz:

$$(1 - x_1)(1 - x_2) \dots (1 - x_n) = n.$$

Xulosa. Maqolada matematikaning turli bo'limlariga tegishli bo'lgan masalalar yechimlari bir necha qiziqarli usullarda berildi. O'rta arifmetik, o'rta geometrik qiymatlar, ko'phadlar va geomerik masalalar ko'rib chiqildi. Bu kabi masalalar

yechimini o‘rganish orqali biz muammolarni hal qilish, mantiqiy fikrlash, va qaror qabul qilish qobiliyatlarimizni rivojlantiradi. Xulosa qilib aytganda ushbu maqoladagi matematik masalalar va ularning yechimlari matematikaga bo‘lgan qiziqishni oshirishga qaratilgan. Umid qilamizki, ushbu masalalarni yechish jarayoni sizga matematika naqadar qiziqarli va foydali ekanligini ko‘rsatdi. Matematikani o‘rganish orqali siz o‘z dunyoqarashingizni kengaytirasiz va yangi imkoniyatlar eshigini ochasiz.

Foydalanilgan adabiyotlar

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