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EYLER INTEGRALLARI BETA VA GAMMA FUNKSIYALARI VA ULAR ORASIDAGI BOG'LANISHLAR.

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Annotatsiya: Ushbu maqolada Eyler integrallarining asosiy turlari va ularning xossalari hamda ular orasidagi bog'lanishlar keltirilgan. Bulardan foydalanib, ba'zi bir xosmas integrallarni hisoblashga oid misollarning yechish metodlari keltirilgan.

Kalit so'zlar: Eyler integrallari, beta funksiya, gamma funksiya, xosmas integral.

KIRISH

Eyler integrallari — matematikada maxsus integrallar turidir. Ular odatda differensial tenglamalarni hal qilishda yoki matematik modellashtirishda keng qo'llaniladi. Eyler integrallari nomi mashxur matematik Leonard Eyler nomi bilan bog'liq bo'lib, asosan 2 xil turi amaliyotda keng qo'llaniladi. Ular odatda "beta" va "gamma" funksiya deb yuritiladi. Ushbu maqolada bu funksiyalarning

ADABIYOTLAR SHARHI

Eyler integrallari, Beta va Gamma funksiyalari matematik tahlilning muhim elementlari bo'lib, ularning tadqiqi matematik fizika, ehtimollar nazariyasi, statistik analiz va muhandislik sohalarida katta ahamiyatga ega. Ushbu funksiyalar va ular bilan bog‘liq integrallarni chuqur o‘rganish uchun ko‘plab ilmiy manbalar va adabiyotlar mavjud.

Leonard Eylerning klassik ishlarida Gamma va Beta funksiyalarining boshlang‘ich g‘oyalari va ularning matematik xossalari batafsil bayon etilgan [1]. Ayniqsa, Gamma funksiyasining “uzluksiz faktorial” sifatida kiritilishi matematik analiz tarixida muhim yangilik bo‘lgan. Gamma va Beta funksiyalari bo‘yicha zamonaviy matematik adabiyotlarda bu funksiyalar bilan bog‘liq ko‘plab yangi natijalar va tadbiqlar taqdim etilgan. Masalan, Abramowitz va Stegunning "Handbook of Mathematical Functions" kitobi ushbu mavzu bo‘yicha keng qamrovli ma’lumotlarni taqdim etadi [2]. Ushbu qo‘llanma matematik tahlilning klassik va zamonaviy natijalarini o‘zida mujassamlashtirgan.

Ilmiy maqolalar So‘nggi yillarda nashr qilingan ilmiy maqolalar Beta va Gamma funksiyalarining ehtimollar nazariyasida, maxsus funksiyalarda va analitik davom ettirish masalalarida qo‘llanilishiga bag‘ishlangan. Xususan, matematik fizika va qatorlarni tahlil qilishda ushbu funksiyalarni qo‘llash bo‘yicha yangi yondashuvlar kiritilgan.

ASOSIY TUSHUNCHALAR VA XOSSALAR

Ta’rif: Ushbu

$\int_0^1 x^{a-1}(1-x)^{b-1}dx$

parametrga bog’liq xosmas integral beta funksiya (1-tur Eyler integrali) deyiladi va $B(a,b)$ kabi belgilanadi:

$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx\,,\quad (a>0, b>0).$

Demak, beta funksiya

$$\{(a,b)\in R^2: a\in(o,\infty), b\in(0,\infty)\}$$

to‘plamda aniqlangan funksiya.

1-teorema. Ushbu

$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$

integral $M_0=\{(a,b)\in R^{2:}: a\in[a_0,+\infty),\ b\in[b_n,+\infty),\ a_0>0, b_0>0\}$

to‘plamda tekis yaqinlashuvchi bo’ladi.

$B(a,b)$ funksiyani ifodalovchi integralni ikki qismga

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx + \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx$$

ko‘rinishda ajratib, har bir integralni tekis yaqinlashishga tekshiramiz.

Parametr $a \geq a_{0,}(a_0 \geq 0), \forall b > 0, \forall x \in \left(0, \frac{1}{2}\right]$ da

$$x^{a-1}(1-x)^{b-1}dx \leq x^{a_0-1}(1-x)^{b-1}dx \leq 2(1-x)^{a_0-1}$$

va $a > 0$ bo‘lganda

$$\int_0^{\frac{1}{2}} x^{a-1}\,dx$$

integralning yaqinlashuvchi bo‘lishidan Veyershtrass alomatiga ko’ra

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx$$

integralning $a \geq a_{0,}(a > 0)$da tekis yaqinlashuvchanligi topamiz. Shuningdek, parametr $b \geq b_0,\ (b_0 > 0),\ \forall a > 0,\ \forall x \in \left(\frac{1}{2}, 1\right]$ da

$$x^{a-1}(1-x)^{b-1}dx \leq x^{a-1}(1-x)^{b_0-1}dx \leq 2(1-x)^{b_0-1}$$

va $b > 0$ bo‘lganda

$$\int_{\frac{1}{2}}^1 (1-x)^{b-1}dx$$

integralning yaqinlashuvchi bo‘lishidan Veyershtrass alomatiga ko‘ra

$$\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx$$

integralning $b \geq b_0,\ (b_0 > 0)$ da tekis yaqinlashuvchi bo‘lishini topamiz. Demak,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integral $M_0 = \{(a,b) \in R^2 : a \in [a_0, +\infty) b \in [b_{0,} +\infty), a_0 > 0, b_0 > 0\}$ to‘plamda tekis yaqinlashuvchi bo‘ladi.

Natija. $B(a,b)$ funksiya

$$M = \{(a,b) \in R^2 : a \in (0,+\infty), b \in (0,+\infty)\}$$

To’plamda uzluksiz bo’ladi.

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Bu tasdiq

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integralning tekis yaqinlashuvchanligidan hamda integral ostidagi funksiyaning M toʻplamda uzluksiz boʻlishidan kelib chiqadi.

$B(a,b)$ funksiyaning xossalari. Endi

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

funksiyalarini xossalarini keltiramiz .

1)$B(a,b)$ funksiya a va b argumentlariga nisbatan simmetrik funksiya, ya'ni:

$$B(a,b) = B(b,a),(a > o,,b > 0)$$

$B(a,b)$ ni ifodalovchi integralda $x = 1 - k$ almashtirish bajarib topamiz:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = -\int_1^0 (1-k)^{a-1}k^{b-1}dk =$$

$$\int_0^1 k^{b-1}(1-k)^{a-1}dk = B(b,a)$$

2) $B(a,b)$ funksiyani quyidagicha ifodalash ham mumkin:

$$B(a,b) = \int_0^{+\infty} \frac{k^{a-1}}{(1+k)^{a+b}}dk \qquad (1)$$

$B(a,b)$ ni ifodalovchi integralda $x = \frac{k}{1+k}$ almashtirish bajarib topamiz:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{+\infty} \left(\frac{k}{1+k}\right)^{a-1}\left(1-\frac{k}{1+k}\right)^{b-1}\frac{dk}{(1+k)^2} =$$

$$\int_0^{+\infty} \frac{k^{a-1}}{(1+k)^{a+b}}dk.$$

Agar (1) da $b = 1 - a, (0 < a < 1)$ deyilsa, u holda

$$B(a,1-a) = \int_0^{+\infty} \frac{k^{a-1}}{1+k}dk = \frac{\pi}{\sin \pi\alpha}$$

boʻladi. Xususan: $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi.$

3) $B(a,b)$ funksiya uchun ushbu

$$B(a+1,b) = \frac{a}{a+b}B(a,b), (a > 0, b > 0)$$

formula oʻrinli bo'ladi.

Ravshanki, $B(a+1,b) = \int_0^1 x^a(1-x)^{b-1}dx.$

Bu integralni boʻlaklab integrallaymiz:

$B(a+1,b)=\int_0^1 x^a(1-x)^{b-1}dx=-\frac{1}{b}\int_0^1 x^a\, d((1-x)^b) \qquad =-\frac{1}{b}x^a(1-$
$x)^b|_0^1+\frac{a}{b}\int_0^1 x^a(1-x)^b dx=\frac{a}{b}\int_0^1 x^{a-1}(1-x)^b dx=$

$$=\frac{a}{b}\left[\int_0^1 x^{a-1}(1-x)^{b-1}dx-\int_0^1 x^a(1-x)^{b-1}dx\right]=\frac{a}{b}B(a,b)--\frac{a}{b}B(a+1,b)$$

Natijada

$$B(a+1,b)=\frac{a}{b}B(a,b)-\frac{a}{b}B(a+1,b)$$

bo‘lib, undan

$$B(a+1,b)=\frac{a}{a+b}B(a,b)$$

bo‘lishi kelib chiqadi.

$B(a,b)$ funksiya simmetrik bo‘lganligidan

$$B(a,b+1)=\frac{b}{a+b}B(a,b) \qquad (3)$$

ifodani yozish mumkin.

Natija. $B(m,n)$ funksiyaga $(m\in N, n\in N)$ (2) va (3) formulalarni takror qo‘llash natijasida

$$B(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

bo‘lishi kelib chiqadi.

Misol: $\int_0^\infty \frac{1}{1+x^5}dx$ hisoblang:

$$\frac{1}{1+x^5}=t,\quad \frac{1}{t}=1+x^5,\quad x^5=\frac{1}{t}-1,\quad x=\left(\frac{1}{t}-1\right)^{\frac{1}{5}}$$

$$t_1=1,\quad t_2=0$$

$$dx=\frac{1}{5}\left(\frac{1}{t}-1\right)^{-\frac{4}{5}}\cdot\left(-\frac{1}{t^2}\right)dt \qquad dx=\frac{-1}{5t^2}\cdot\left(\frac{1}{t}-1\right)^{-\frac{4}{5}}dt$$

$$\int_0^\infty \frac{1}{1+x^5}dx=\int_1^0 t\cdot\frac{-1}{5t^2}\cdot\left(\frac{1}{t}-1\right)^{-\frac{4}{5}}dt=\int_1^0\frac{-1}{5t}\cdot\left(\frac{1}{t}-1\right)^{-\frac{4}{5}}dt$$

$$\frac{1}{5}\int_0^1 t^{-1}\cdot\left(\frac{1-t}{t}\right)^{-\frac{4}{5}}dt=\frac{1}{5}\int_0^1 t^{-\frac{1}{5}}\cdot(1-t)^{-\frac{4}{5}}dt$$

$$\frac{1}{5}\int_0^1 t^{\frac{4}{5}-1}\cdot(1-t)^{\frac{1}{5}-1}dt$$

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

demak,

$$\frac{1}{5}\int_0^1 t^{\frac{4}{5}-1}\cdot(1-t)^{\frac{1}{5}-1}dt = \frac{1}{5}B\left(\frac{4}{5},\frac{1}{5}\right)$$

$$B(a,1-a) = \frac{\pi}{\sin\pi a}$$

xossaga ko‘ra,

$$\frac{1}{5}B\left(\frac{4}{5},\frac{1}{5}\right) = \frac{\pi}{5\sin(\pi\cdot\frac{4}{5})} = \frac{\pi}{5\sin(\frac{\pi}{5})}.$$

Endi biz $\sin(\frac{\pi}{5})$ ni hisoblab qo‘yamiz

$$\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5},\ \sin 2\theta = \sin 3\theta$$

$$2\sin\theta\cos\theta = 3\sin\theta - 4\sin^3\theta \qquad 2\cos\theta = 3 - 4\sin^2\theta$$

$$2\cos\theta = 4\cos^2\theta - 1 \qquad 4\cos^2\theta - 2\cos\theta - 1 = 0 \quad 4t^2 - 2t - 1 = 0$$

$$D = \sqrt{4+16} = 2\sqrt{5} \quad t_{1,2} = \frac{2\pm2\sqrt{5}}{8} \quad t_{1,2} = \frac{1\pm\sqrt{5}}{4}$$

$$\cos\theta = \frac{1+\sqrt{5}}{4} \quad \sin^2\theta = 1-\cos^2\theta \quad \sin^2\theta = 1 - \frac{1+2\sqrt{5}+5}{16} \quad \sin^2\theta = \frac{10-2\sqrt{5}}{16}$$

$$\sin\theta = \frac{\sqrt{10-2\sqrt{5}}}{4} \quad \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

demak,

$$\frac{1}{5}B\left(\frac{4}{5},\frac{1}{5}\right) = \frac{4\pi}{5\sqrt{10-2\sqrt{5}}} = \frac{4\pi\sqrt{10+2\sqrt{5}}}{5\sqrt{80}} = \frac{4\pi\sqrt{10+2\sqrt{5}}}{5\cdot4\sqrt{5}} = \frac{\pi\sqrt{10+2\sqrt{5}}}{5\sqrt{5}}$$

Javob: $\int_0^\infty \frac{1}{1+x^5}dx = \frac{\pi\sqrt{10+2\sqrt{5}}}{5\sqrt{5}}$.

2. **Misol:** Isbotlang: $\int_0^1(-\ln x)^n dx = n!$

$$\int_0^1 \frac{x(-\ln x)^n dx}{x} = \int_0^1 x(-\ln x)^n d(\ln x) = -\int_0^1 x(-\ln x)^n d(-\ln x)$$

$$\begin{cases} x_1 = 0, x_2 = 1 \\ -\ln x = t,\ x = e^{-t} \\ t_1 = -\ln 0 = +\infty,\ t_2 = -\ln 1 = 0 \end{cases}$$

$$-\int_{+\infty}^{0} t^n e^{-t}dt = \int_0^{+\infty} t^n e^{-t}dt = \int_0^{+\infty} t^{n+1-1}e^{-t}dt$$

Gamma funksiyasiga ko‘ra,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx, \qquad n\in N,\ \Gamma(n+1) = n!$$

$$\int_0^{+\infty} t^{n+1-1}e^{-t}dt = \Gamma(n+1) = n!$$

Javob: $\int_0^1 (-\ln x)^n dx = n!$.

MUHOKAMA

Beta va Gamma funksiyalari nafaqat nazariy jihatdan muhim balki amaliy tadqiqotlar uchun ham zarur hisoblanadi. Gamma funksiyasining uzluksiz faktorial sifatida ishlatilishi matematik analiz sohasidagi ko‘plab muammolarni yechishda muhim ahamiyat kasb etadi. Beta funksiyasi esa ko‘p hollarda qatorlar va integrallarni soddalashtirishda qulay vosita bo‘lib xizmat qiladi.

Shuningdek, ilmiy maqolalar va zamonaviy tadqiqotlar ushbu funksiyalarning ehtimollar nazariyasi va matematik fizika masalalaridagi ahamiyatini yanada yoritmoqda. Dasturiy vositalar orqali bu funksiyalar bilan ishlashning osonlashgani murakkab matematik hisob-kitoblarni bajarish imkonini yaratdi.

Kelgusida ushbu funksiyalarning yangi qo‘llanilish sohalarini kashf etish va ularni yanada chuqurroq o‘rganish matematik tadqiqotlarning muhim yo‘nalishlaridan biri bo‘lib qolishi kutilmoqda.

XULOSA

Beta va gamma funksiyalari matematikada o‘ta muhim o‘rin tutuvchi vositalar bo‘lib, ular Eyler integralini hisoblashda asosiy ahamiyat kasb etadi. Ushbu funksiyalar murakkab integrallarni soddalashtirishda va xosmas integrallarni yechishda qulaylik yaratadi. Beta va gamma funksiyalarining o‘zaro bog‘liqligi va xossalari orqali nafaqat matematik nazariyani chuqurroq o‘rganish, balki amaliy masalalarni samarali yechish ham mumkin. Ular matematik modellashtirish, fizika, texnika va iqtisodiyot kabi ko‘plab sohalarda qo‘llaniladi va muhim ilmiy asos bo‘lib xizmat qiladi.

ADABIYOTLAR RO`YXATI

1. Eyler, L. Institutiones Calculi Integralis, 1768.
2. Abramowitz, M., & Stegun, I. A., Handbook of Mathematical Functions, Dover Publications, 1964

3. Архипов Г.И., Садовничий В.А., Чубариков В.Н. Лекции по математическому анализу / Под ред. В. А. Садовничего. — М.: Высшая школа, 1999. — С. 43—44. — ISBN 5-06-003596-4.

4. Xakimov, R. M. (2019). IMPROVEMENT OF ONE RESULT FOR THE POTTS MODEL ON THE CALEY TREE. Scientific and Technical Journal of Namangan Institute of Engineering and Technology, 1(6), 3-8.

5. Umirzaqova, K. O. (2020). PERIODIC GIBBS MEASURES FOR HARD-CORE MODEL. Scientific Bulletin of Namangan State University, 2(3), 67-73.

6. Уктамалиев, И. К. (2022). О предгеометриях конечно порожденных коммутативных полугрупп. In МАЛЬЦЕВСКИЕ ЧТЕНИЯ (pp. 166-166).

7. Уктамалиев, И. К. (2022). О числе счётных моделей аддитивной теории натуральных чисел.

8. O‘G, O. K. I. Q., O’G’Li, J. A. H., & O‘G, H. T. X. D. (2024). FUNKSIONAL QATORNI HADLAB INTEGRALLASH VA DIFFERENSIALLASHDAN FOYDALANIB BA’ZI BIR SONLI QATORLAR YIG ‘INDISINI TOPISH METODLARI. Science and innovation, 3(Special Issue 57), 411-416.

9. O‘G, O. K. I. Q., Qizi, N. M. S. N., & Qizi, A. M. O. A. (2024). TEYLOR QATORI YORDAMIDA BA’ZI BIR SONLI QATORLARNING YIG ‘INDISINI TOPISH USULLARI. Science and innovation, 3(Special Issue 57), 275-277.

10. Xo‘jamqulov, R. (2024). Matematika fanini o ‘rganishda Maple platformasidan foydalanish imkoniyatlari va amaliy jihatlari. Universal xalqaro ilmiy jurnal, 1(12), 335-338.