TÓRTINCHI TARTIBLI TENGLAMA UCHUN BIR CHEGARAVIY MASALANING YECHILISHI

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Annotatsiya: Maqolada tórtinchi tartibli tartibli xususiy xosilali differencial tenglama uchun tórtburchakli sohada bir aralash-chegaraviy masala kórib chiqildi. Masala yechimining mavjudligi va yagonaligi kórsatildi. Masala yechimini topish uchun ózgaruvchilarni ajratish usuli qóllanildi. Masala yechimi xususiy funkciyalar bóyicha qatorga yiǵib topildi. Yechimning yagonaligi ortonormallangan sistemaning tóliqligidan foydalanib kórsatildi.

Kalit sózlar: boshlanǵich-chegaraviy masala, ózgaruvchilarni ajratish usuli, yechimning yagonaligi, mavjudligi, xususiy qiymatlar va funkciyalar.

Kirish: Geofizikaning, okeanologiyaning birqancha masalalari, texnikada kriogen suyuqliklarni shuningdek, stratificirlangan suyuqliklarni foydalanish masalalari, elastiklik teoriyasining masalalari, bir jinsli sterejenning yoki balkaning tebranish tenglamasi tórtinchi tartbli xususiy xosilali differencial tenglamalar uchun qóyilgan chegaraviy, boshlanǵich-chegaraviy masalalarni yechishga keltiriladi.

Tórtburchakli sohada tórtinchi tartibli xususiy xosilali differencial tenglamalar uchun qóyilgan masalalar M.M. Smirnovning[10], T.D. Djuraevning va A. Sopuevning[6], K.B. Sabitovning [8-9], A.Berdishevning va B.J. Kadirkulovning[4], Ya. Megralievning [7], D. Amanovning[1-2], T. Yuldashovning[12], A.Urinovning[11], Yu. Apakovning[3] va h.k. ishlarida qaralgan.

[10] monografiyasida tórtinchi tártibli xususiy xosilali differencial tenglamalarning tóliq klassifikaciyasi va u tenglamalar uchun qóyilgan masalalarni yechish qaralgan. [5] ishda tórtinchi tartibli tenglamalar uchun bir chegaraviy masala qaralgan.

Masalaning qóyilishi

$\Omega=\{(x,y):0<x<p,0<t<T\}$ sohada

$$U_{ttt}-U_{xxxx}=f(x,t) \tag{1}$$

tenglamani kóramiz, bu yerda p,T$\in$R.

Masala 1. Ω sohada (1) tenglamaning quyidagi

$$u\big|_{x=0}=0,\ u\big|_{x=p}=0,\ u_{xx}\big|_{x=0}=0,\ u_{xx}\big|_{x=p}=0,\quad 0\le x\le p \tag{2}$$

$$u\big|_{t=0}=\psi_1(x) \quad u_t\big|_{t=0}=\psi_2(x) \quad u_t\big|_{t=T}=\psi_3(x) \quad 0\le t\le T \tag{3}$$

chegaraviy shartlarni qanoatlandiradigan

$$u(x,y)\in C_{x,t}^{4,3}(\Omega)\cap C_{x,t}^{2,1}(\bar{\Omega})$$

sinfga tegishli yechimini izlaymiz, bunda $\psi_i(x), i=\overline{1,3}$ yetarlicha silliq funkciyalar va

$$\psi_1(0)=\psi_1(p)=\psi_1''(0)=\psi_1''(p)=0, \qquad \psi_i(0)=\psi_i(p)=0, \quad \text{i=2,3.} \tag{4}$$

2. Masala yechimining mavjudligi

Masalaning yechimga ega ekanligini isbotlaymiz: (1) tenglamaning (2) chegaraviy shartlarni qanoatlandiradigan trivial bólmagan yechimini

$$u(x,y)=X(x)U(t). \tag{5}$$

kórinishida izlaymiz.

(5) ni (1) tenglamaga qóyish va ózgaruvchilarni ajratish orqali X(x) va U(t) funkciyalar uchun quyidagi differencial tenglamalarga ega bólamiz:

$$X^{(4)}(x)-\lambda^4 X(x)=0 \tag{6}$$

$$U_k^{(3)}(t)-\lambda_k^{\,4}U_k(t)=f_k(t) \tag{7}$$

bunda λ^4 ózgarmas.

(2) chegaraviy shartlarni hisobga olgan holda (6) tenglama uchun quyidagi masalaga ega bólamiz:

$$\begin{cases} X^{(4)}(x)-\lambda^4 X(x)=0 \\ X(0)=X(p)=X''(0)=X''(p)=0 \end{cases} \tag{8}$$

(8) masala trivial bólmagan yechimga ega bóladi, bunda

$$\lambda_k^4=\left(\frac{\pi k}{p}\right)^4, \quad n=1,2,3,\dots \ .$$

Bu sonlar (8) masalaning xususiy qiymatlari deyiladi va ularga mos keladigan normalangan xususiy funkciyalar quyidagi kórinishda bóladi:

$$X_k(x)=\sqrt{\frac{2}{p}}\sin\frac{\pi k}{p}x. \tag{9}$$

(7) ning bir jinsli yechimi

$$U_k(t)=a_k e^{\nu_k t}+e^{-\frac{1}{2}\nu_k t}\left(b_k\cos\left(\frac{\sqrt{3}}{2}\nu_k t\right)+c_k\sin\left(\frac{\sqrt{3}}{2}\nu_k t\right)\right)$$

kórinishida bóladi, bunda $\nu_k = \sqrt[3]{\lambda_k^4} = \left(\frac{\pi k}{p}\right)^{4/3}$, $n \in N$ va a_k, b_k, c_k - hozircha nomalum ózgarmaslar. Ózgarmaslarni variaciyalash usulidan foydalanib yechimni

$$U_k(t) = a_k(t)e^{\nu_k t} + e^{-\frac{1}{2}\nu_k t}\left(b_k(t)\cos\left(\frac{\sqrt{3}}{2}\nu_k t\right) + c_k(t)\sin\left(\frac{\sqrt{3}}{2}\nu_k t\right)\right), \tag{10}$$

kórinishida izlaymiz.

(1) tenglamaning umumiy yechimi

$$u(x,y) = \sqrt{\frac{2}{p}}\sum_{k=1}^{\infty}\left(a_k(t)e^{\nu_k t} + e^{-\frac{1}{2}\nu_k t}\left(b_k(t)\cos\left(\frac{\sqrt{3}}{2}\nu_k t\right) + c_k(t)\sin\left(\frac{\sqrt{3}}{2}\nu_k t\right)\right)\right)\sin\frac{\pi k}{p}x$$

qator kórinishida bóladi. $a_k(t), b_k(t), c_k(t)$ funkciyalarni topish uchun (10) tengliktan 3 marta xosila olib (1) tenglamaga qóyamiz va quyidagi tenglamalar sistemasiga ega bólamiz:

$$a_k'(t)e^{\nu_k t} + e^{-\frac{1}{2}\nu_k t}\left[b_k'(t)\cos\frac{\sqrt{3}}{2}\nu_k t + c_k'(t)\sin\frac{\sqrt{3}}{2}\nu_k t\right] = 0$$

$$a_k'(t)e^{\nu_k t} + e^{-\frac{1}{2}\nu_k t}\left(\left[-\frac{1}{2}\cos\frac{\sqrt{3}}{2}\nu_k t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}\nu_k t\right]b_k'(t) + \right.$$
$$\left. + \left[-\frac{1}{2}\sin\frac{\sqrt{3}}{2}\nu_k t + \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}\nu_k t\right]c_k'(t)\right) = 0$$

$$a_k'(t)e^{\nu_k t} + e^{-\frac{1}{2}\nu_k t}\left(\left[-\frac{1}{2}\cos\frac{\sqrt{3}}{2}\nu_k t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}\nu_k t\right]b_k'(t) + \right.$$
$$\left. + \left[-\frac{1}{2}\sin\frac{\sqrt{3}}{2}\nu_k t - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}\nu_k t\right]c_k'(t)\right) = \frac{1}{\nu_k^2}f_k(t)$$

Bu sistemani yechish orqali $a_k'(t), b_k'(t), c_k'(t)$ lar uchun

$$a_k'(t) = \frac{e^{-\nu_k t}}{\nu_k^2}f_k(t)$$

$$b_k'(t) = -\frac{2}{\sqrt{3}}\frac{e^{\frac{1}{2}\mu_k t}}{\nu_k^2}f_k(t)\sin\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\nu_k t\right)$$

$$c_k'(t) = -\frac{2}{\sqrt{3}}\frac{e^{\frac{1}{2}\nu_k t}}{\nu_k^2}f_k(t)\cos\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\nu_k t\right)$$

tengliklarga ega bólamiz. Endi bularni integrallash orqali $a_k(t), b_k(t), c_k(t)$ funkciyalarni topamiz:

$$a_k(t)=a_k+\frac{1}{\nu_k^2}\int_0^t f_k(\tau)e^{-\nu_k\tau}d\tau$$

$$b_k(t)=b_k-\frac{2}{\sqrt{3}\nu_k^2}\int_0^t f_k(\tau)e^{\frac{1}{2}\nu_k\tau}\sin\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\nu_k\tau\right)d\tau$$

$$c_k(t)=c_k-\frac{2}{\sqrt{3}\nu_k^2}\int_0^t f_k(\tau)e^{\frac{1}{2}\nu_k\tau}\cos\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\nu_k\tau\right)d\tau$$

Bu topilganlarni (10) tenglikka qóyish orqali

$$U_k(t)=a_ke^{\nu_k t}+e^{-\frac{1}{2}\nu_k t}\left(b_k\cos\left(\frac{\sqrt{3}}{2}\nu_k t\right)+c_k\sin\left(\frac{\sqrt{3}}{2}\nu_k t\right)\right)+$$

$$+\frac{1}{\nu_k^2}\int_0^t f_k(\tau)e^{-\nu_k(t-\tau)}d\tau-\frac{2}{\sqrt{3}\nu_k^2}\int_0^t f_k(\tau)e^{-\frac{1}{2}\nu_k(t-\tau)}\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_k(t-\tau)\right)d\tau \qquad (12)$$

yechimga ega bólamiz. Bunda a_k, b_k, c_k -hozircha nomalum ózgarmaslar. a_k, b_k, c_k ózgarmaslarni topish uchun (3) shartlardan foydalanamiz. (3) shartlardan

$$U_k(0)=\psi_{1k},\ U_k'(0)=\psi_{2k},\ U_k'(T)=\psi_{3k}$$

kelib chiqadi, bu yerda

$$\psi_i(x)=\sqrt{\frac{2}{p}}\sum_{k=1}^{\infty}\psi_{ik}\sin\frac{\pi k}{p}x,\ i=1,2,3$$

$$\psi_{ik}=\sqrt{\frac{2}{p}}\int_0^p\psi_i(\xi)\sin\frac{\pi k}{p}\xi d\xi,\qquad i=1,2,3. \qquad (13)$$

(10) funkciyani (13) shartlarga qóyib

$$\begin{cases} a_k+b_k=\psi_{1k}, \\ \nu_ka_k+\frac{1}{2}\nu_kb_k+\frac{\sqrt{3}}{2}\nu_kc_k=\psi_{2k}, \\ \nu_ke^{\nu_kT}a_k+\nu_ke^{\frac{1}{2}\nu_kT}\left[\cos\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)b_k+\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)c_k\right]+ \\ +\frac{1}{\nu_k}\int_0^T f_k(\tau)e^{\nu_k(T-\tau)}d\tau+\frac{2}{\sqrt{3}\nu_k}\int_0^T f_k(\tau)e^{-\frac{1}{2}\nu_k(T-\tau)}\sin\left(\frac{\sqrt{3}}{2}\nu_k(T-\tau)\right)d\tau=\psi_{3k} \end{cases} \qquad (14)$$

tenglamalar sistemasiga ega bólamiz.

(14) sistemaning determinanti

$$\Delta=\nu_k^2e^{\frac{1}{2}\nu_kT}\sin\frac{\sqrt{3}}{2}\nu_kT-\nu_k^2e^{\frac{1}{2}\nu_kT}\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)+\frac{\sqrt{3}}{2}\nu_k^2e^{\nu_kT}$$

bóladi. Bu determinant qiymati 0 dan katta bóladi.

Unda (14) sistemaning yechimi

$$a_k=\frac{1}{\Delta}\left[\left[\psi_{1k}\nu_k^2e^{\frac{1}{2}\nu_kT}\sin\frac{\sqrt{3}}{2}\nu_kT-\psi_{2k}\nu_ke^{\nu_kT}\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)+\frac{\sqrt{3}}{2}\nu_k\tilde{\psi}_{3k}\right],\right.$$

$$b_k=\frac{1}{\Delta}\left[-\psi_{1k}\left(\nu_k^2e^{\frac{1}{2}\nu_kT}\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)-\frac{\sqrt{3}}{2}\nu_k^2e^{\nu_kT}\right)+\right.$$

$$\left.+\psi_{2k}\nu_ke^{\frac{1}{2}\nu_kT}\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)-\frac{\sqrt{3}}{2}\nu_k\tilde{\psi}_{3k}\right] \qquad (15)$$

$$c_k=\frac{1}{\Delta}\left[\psi_{1k}\left(\nu_k^2e^{\frac{1}{2}\nu_kT}\cos\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)-\frac{1}{2}\nu_k^2e^{\nu_kT}\right)-\right.$$

$$\left.-\psi_{2k}\left(\nu_ke^{\frac{1}{2}\nu_kT}\cos\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\nu_kT\right)-\nu_ke^{\nu_kT}\right)-\frac{1}{2}\nu_k\tilde{\psi}_{3k}\right]$$

bóladi, bu yerda

$$\tilde{\psi}_{3k}=\psi_{3k}-\frac{1}{\nu_k}\int_0^T f_k(\tau)e^{\nu_k(T-\tau)}d\tau-\frac{2}{\sqrt{3}\nu_k}\int_0^T f_k(\tau)e^{-\frac{1}{2}\nu_k(T-\tau)}\sin\left(\frac{\sqrt{3}}{2}\nu_k(T-\tau)\right)d\tau$$

a_k, b_k, c_k koefficientlarning topilgan qiymatlarini (12) da órniga oborib qóyamiz. Demak (1) tenglamaning (2) va (3) shartlarni qanoatlandiradigan yechimi (12) kórinishida bóladi. a_k, b_k, c_k koefficientlarini baholaymiz:

$$|a_k|\le\frac{C_1}{\Delta}\left[|\psi_{1k}|\nu_k^2e^{\frac{1}{2}\nu_kT}+|\psi_{2k}|\nu_ke^{\nu_kT}+|\psi_{3k}|\nu_k\right]$$

$$|b_k|\le\frac{C_2}{\Delta}\left[|\psi_{1k}|\nu_k^2e^{\nu_kT}+|\psi_{2k}|\nu_ke^{\frac{1}{2}\nu_kT}+|\psi_{3k}|\nu_k\right] \qquad (16)$$

$$|c_k|\le\frac{C_3}{\Delta}\left[|\psi_{1k}|\nu_k^2e^{\nu_kT}+|\psi_{2k}|\nu_ke^{\nu_kT}+|\psi_{3k}|\nu_k\right].$$

$\psi_1(x)$ ni 5 marta va $\psi_i(x),\ i=2,3$ ni 4 marta bólaklab integrallaymiz.

$$\psi_{1k}=\frac{1}{\lambda_k^5}\bar{\psi}_{1k}^{(5)},\qquad \bar{\psi}_{ik}^{(5)}=\sqrt{\frac{2}{p}}\int_0^p\psi_{1k}^{(5)}(x)\cos\lambda_k x dx,$$

$$\psi_{ik}=\frac{1}{\lambda_k^4}\bar{\psi}_{ik}^{(4)},\qquad \bar{\psi}_{ik}^{(4)}=\sqrt{\frac{2}{p}}\int_0^p\psi_{ik}^{(4)}(x)\sin\lambda_k x dx,\quad i=2,3.$$

bu yerda $\psi_1^{(2j)}(0)=\psi_1^{(2j)}(p)=0,\ j=0,1,2,\qquad \psi_i^{(2j)}(0)=\psi_i^{(2j)}(p)=0,\ j=0,1;$ i=2,3.

ψ_{ik} larning bu qiymatlarini (16) tengsizlikning óng tarafiga qóyib

$$|\psi_{1k}|\le M\frac{\left|\bar{\psi}_{1k}^{(5)}\right|}{k^5},\quad |\psi_{ik}|\le M\frac{\left|\bar{\psi}_{1k}^{(4)}\right|}{k^4},\quad i=2,3. \tag{17}$$

a_k, b_k, c_k uchun biz quyidagi baholashlarni yoza olamiz:

$$|a_k|\le M\left(\frac{\left|\bar{\psi}_{1k}^{(5)}\right|}{k^5}+\frac{\left|\bar{\psi}_{2k}^{(4)}\right|}{k^5}+\frac{\left|\bar{\psi}_{3k}^{(4)}\right|}{k^5}\right),$$

$$|b_k|\le M\left(\frac{\left|\bar{\psi}_{1k}^{(5)}\right|}{k^5}+\frac{\left|\bar{\psi}_{2k}^{(4)}\right|}{k^5}+\frac{\left|\bar{\psi}_{3k}^{(4)}\right|}{k^5}\right),$$

$$|c_k|\le M\left(\frac{\left|\bar{\psi}_{1k}^{(5)}\right|}{k^5}+\frac{\left|\bar{\psi}_{2k}^{(4)}\right|}{k^5}+\frac{\left|\bar{\psi}_{3k}^{(4)}\right|}{k^5}\right)$$

(11) qatorni hadma-had t bóyicha 3 marta, x bóyicha 4 marta differenciallaymiz. (11) qatorning va uning mos xosilalaridan tuzilgan qatorlarning $\bar{\Omega}$ sohada teng ólchovli yaqinlashuvchiligini kórsatish kerak.

1-teorema. Eger $\psi_1(x)\in C^5[0,p],\ \ \psi_i(x)\in C^4[0,p], i=2,3$ hamda (4) maxsus shartlarni qanoatlandirsa, unda 1-masala yechimi bor va u (11) qator kórinishida yoziladi.

Isbot. Agar (11) qator va uning u_{xxxx}, u_{ttt} xosilalari $\bar{\Omega}$ sohada bir qiymatli aniqlansa, unda ushbu qator orqali berilgan $u(x,y)$ funkciya 1 masala yechimi bóladi.

(11) dan

$$|u(x,y)|\le\sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\left(|a_k|+|b_k|+|c_k|\right) \tag{18}$$

kelib chiqadi. Unda (15) ni hisobga olgan holda (16) dan

$$|u(x,y)| \le M\left(\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{1k}^{(5)}\right|}{k^5}+\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{2k}^{(5)}\right|}{k^5}+\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{3k}^{(5)}\right|}{k^5}\right)<\infty$$

bóladi.

$$u_{xxxx}=\sum_{n=1}^{\infty}\lambda_k^4\left[a_k e^{\nu_k y}+e^{-\frac{1}{2}\nu_k y}\left(b_k\cos\left(\frac{\sqrt{3}}{2}\nu_k y\right)+c_k\sin\left(\frac{\sqrt{3}}{2}\nu_k y\right)\right)\right]X_k(x)$$

$$|u_{xxxx}| \le \sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\lambda_k^4\left(|C_1|e^{\nu_k q}+|C_2|+|C_3|\right)\le M\left(\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{1k}^{5}\right|}{k}+\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{2k}^{5}\right|}{k}+\sum_{n=1}^{\infty}\frac{\left|\bar{\psi}_{3k}^{5}\right|}{k}\right).$$

Bu tengsizlikning óng tarafiga Koshi-Bunyakovskiy tengsizligini qóllaymiz:

$$|u_{ttt}| \le M\left(\sqrt{\sum_{n=1}^{\infty}\left|\bar{\psi}_{1k}^{(5)}\right|^2}+\sqrt{\sum_{n=1}^{\infty}\left|\bar{\psi}_{2k}^{(4)}\right|^2}+\sqrt{\sum_{n=1}^{\infty}\left|\bar{\psi}_{3k}^{(4)}\right|^2}\right)\sqrt{\sum_{n=1}^{\infty}\frac{1}{k^2}}\le$$

$$\le M\left(\left\|\psi_1^{V}\right\|_{L_2(0,p)}+\left\|\psi_2^{IV}\right\|_{L_2(0,p)}+\left\|\psi_3^{IV}\right\|_{L_2(0,p)}\right)<\infty$$

(11) qatordan t bóyicha uch marta xosila olingan qatorning ham yaqinlashuvchiligi shunday kórsatiladi.

Masala yechimining turǵunligi

Quyidagi normalarni kiritamiz:

$$\left\|u(x,t)\right\|_{L_2[0,p]}=\left(\int_0^p\left|u(x,t)\right|^2dx\right)^{\frac{1}{2}},\ \left\|u(x,t)\right\|_{C(\bar{\Omega})}=\max_{\bar{\Omega}}\left|u(x,t)\right|.$$

2-teorema. Mayli 1-teoremaning shartlari órinli bólsin. Unda 1 masalaning (11) yechimi uchun quyidagi baholashlar órinli bóladi:

$$\left\|u(x,t)\right\|_{L_2[0,p]}\le C_4\left[\left\|\psi_1\right\|_{L_2}+\left\|\psi_2\right\|_{L_2}+\left\|\psi_3\right\|_{L_2}\right],$$

$$\left\|u(x,t)\right\|_{C(\bar{\Omega})}\le C_5\left[\left\|\psi_1\right\|_{W_2^1}+\left\|\psi_2\right\|_{L_2}+\left\|\psi_3\right\|_{L_2}\right].$$

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