RATSIONAL FUNKSIYALARNI INTEGRALLASH

Samarqand davlat pedagogika instituti
talabasi ***Axmatova Mahliyo***

Annotatsiya. Ushbu maqolada ratsional funksiyalarni integrallashning bir necha usullari keltilgan. Xususan noma’lum koeffitsiyentlar, Xevisayd usuli, differensiallashdan foydalanish, Gorner sxemasi va usuli keltirilgan. Bir nechta ratsional kasrlarni ushbu usullardan foydalanib integralini topish masalalari ko‘rib chiqilgan.

Kalit so‘zlar. Ratsional, kasr ratsional, integral, differensial, Gorner sxemasi, Xevisayd usuli, Ostragradskiy usuli.

ИНТЕГРИРОВАНИЕ РАЦИОНАЛЬНЫХ ФУНКЦИЙ

Аннотация. В данной статье приведены несколько методов интегрирования рациональных функций. В частности, представлены методы неопределённых коэффициентов, метод Хевисайда, использование дифференцирования, схема Гонернера и другие. Рассмотрены задачи нахождения интегралов нескольких рациональных дробей с использованием данных методов.

Ключевые слова.Рациональный, рациональная дробь, интеграл, дифференциал, схема Горнера, метод Хевисайда, метод Остроградского.

INTEGRATING RATSIONAL FUNCTIONS

Annotation. This article presents several methods for integrating rational functions. In particular, the methods of undetermined coefficients, the Heaviside cover-up method, using differentiation, Horner’s scheme and others are provided. Problems of finding the integrals of several rational fractions using these methods are considered.

Keywords. Rational, rational fraction, integral, differential, Horner’s scheme, Heaviside cover-up method, Ostrogradsky method.

No’malum koeffitsiyentlar usuli. Ikkita algebraic ko‘phadlarning nisbatiga, ya’ni

$$f(x)=\frac{P_m(x)}{Q_n(x)} \qquad (1)$$

Ifodali ratsional funksiya yoki ratsional kasr deb ataladi. Bunda, $P_m(x)=b_0+b_1x+...+b_mx^m$ va $Q_n(x)=a_0+a_1x+...+b_nx^n$ $(b_m,a_n\neq 0, m\geq 0, n\geq 1)$ haqiqiy koeffitsiyentli ko'phadlar , deb faraz qilinadi.

Agar $m<n$ bo'lsa, u holda $f(x)=\frac{P_m(x)}{Q_n(x)}$ to'g'ri kasr ratsional funksiya, $m\geq n$ bo'lganda esa, noto'g'ri kasr ratsional funksiya deyiladi. Agar (1) ratsional kasr , noto'g'ri kasr bo'lsa u kasrning suratini mahrajiga bo'lish yo'li bilan,

$$f(x)=w+\frac{P_k(x)}{Q_n(x)} \quad (k<n) \qquad (2)$$

ko'rinishga keltiriladi, bunda $w(x)$- biror ko'phad.

Oliy algebra kursidan ma'lumki, har qanday $Q_n(x)$ ko'phadni, ushbu

$$Q_n(x)=a_n(x-\alpha)(x-\beta)...(x-\nu) \qquad (3)$$

ko'rinishda tasvirlash mumkin.(bunda $\alpha,\beta,\ldots,\nu$-lar $Q_n(x)=0$ tenglamaning ildizlari. Agarda ko'phadning ildizlari ichida o'zaro tenglari bo'lsa, u holda, ko'phad,

$$Q_n(x)=a_n(x-\alpha)^r(x-\beta)^s...(x-\nu)^t \qquad (4)$$

ko'rinishga keltiriladi, bunda r,s,…,t – butun sonlar, $\alpha,\beta,\ldots,\nu$ sonlar esa, mos ravishda, $Q_n(x)$ ko'phadning r,s,…,t – karrali ildizlari deyiladi va $r+s+...+t=n$ bo'ladi.

Ko'phadning (3) dagi ildizlari Ichida kompleks ildizlar ham bo'lishi mumkin. Algebra kursidan ma'lumki, agar $z=a+ib$ haqiqiy koeffitsiyentli ko'phadning r-karrali ildizi bo'lsa, u holda unga qo'shma $\bar{z}=a-ib$ son ham ko'phadning r – karrali ildizi bo'ladi. Boshqacha aytganda, agar (4) ning tarkibidagi $(x-\alpha)^r$ va $(x-\bar{\alpha})^r$ larning ko'paytmasi, quyidagicha bo'ladi: $(x-\alpha)^r(x-\bar{\alpha})^r=$

$$=\{[x-(a+ib)][x-(a-ib)]\}^r=\left[x^2-x(a+ib)-x(a-ib)+a^2+b^2\right]=\left[x^2-2ax+a^2+b^2\right]=(x^2+2px+q)^r$$

Bunda $p=-a, q=a^2+b^2. p^2-q<0$ va $p,q\in R$

Xuddi shunday yuqoridagi mulohazalarni boshqa kompleks ildizlar uchun ham yuritsak, u holda (4) quyidagi ko'rinishni oladi:

$$Q_n(x)=A(x-\alpha)^r(x-\beta)^s...(x^2+2px+q)^t(x \ +2ux+v),...,$$

Bunda $\alpha,\beta,..., p,q,u,v-$haqiqiy sonlar, $r,s,...,t,k$ -natural sonlar.

Quyidagi teoremani qaraylik,

Teorema. Agar $\frac{P_m(x)}{Q_n(x)}$ to'g'ri ratsional kasr tarkibidagi $Q_n(x)$ ko'phad (5) shaklda tasvirlangan bo'lsa, u holda, ratsional kasr, yagona ravishda,

$$\frac{P_m(x)}{Q_n(x)}=\frac{A_1}{(x-\alpha)}+\frac{A_2}{(x-\alpha)^2}+...+\frac{A_r}{(x-\alpha)^r}+...+\frac{M_1x+N_1}{x^2+2px+q}+\frac{M_2x+N_2}{(x^2+2px+q)^2}+...+\frac{M_tx+N_t}{(x^2+2px+q)^t}+..$$

(6)

(bunda, A_1, A_2,…, A_r, M_1, N_1, M_2, N_2,…, M_t, N_t,…, - no'malum haqiqiy sonlar) ko'rinishida tasvirlanadi. (6) tenglik, x ning $Q_n(x)$ ko'phadning haqiqiy ildizlariga teng bo'lmagan hamma qiymatlarida o'rinli.

(6) dagi no'malum koeffitsiyentlarni toppish uchun (6) ni umumiy mahrajga keltirib, (umumiy mahraj $Q_n(x)$) ikki ko'phadning tengligi haqidagi teoremaga asosan, o'ng tomon suratida hosil bo'lgan ko'phad bilan $P_m(x)$ ko'phaddagi x ning bir xil darajalari oldidagi koeffitsiyentlarni tenglashtirish natijasida, no'malum koeffitsiyentlarga nisbatan chiziqli algebraik tenglamalar sistemasi hosil bo'ladi. Bu sistemadan no'malum koeffitsiyentlarni topib, topilgan qiymatlarni (6) tenglikka keltirib qo'yamiz. Kasrning yoyilmasidagi no'malum koeffitsiyentlarni topishning bu usuli, no'malum koeffitsiyentlar usuli deb ataladi.

Shunday qilib, $f(x)=\frac{P_m(x)}{Q_n(x)}$ ratsional kasrning integralini hisoblash, $w(x)=c_0x^k+c_1x^{k-1}+...+c_k$ shakldagi ko'phadni integrallashga va quyidagi

$$I.\frac{A}{x-\alpha}, \quad II.\frac{A}{(x-\alpha)^r}, \quad III.\frac{Mx+N}{x^2+px+q}, \quad IV.\frac{Mx+N}{(x^2+px+q)^r} \qquad (7)$$

$(r>1)$ (bunda A, M, N, α, p, q - haqiqiy sonlar, $q-\frac{p^2}{4}>0$) ko'rinishidagi sodda kasrlarni integrallashga keltiriladi. Bu sodda kasrlarning integrallari quyidagicha hisoblanadi.

Quyidagi, $\frac{A}{(x-a)^m}, \frac{Bx+C}{(x^2+px+q)^k}$ $(A, B, C, a, p, q \in R, k, m \in N)$

Ko'rinishidagi sodda kasrlarni qaraymiz. U holda:

1) $m=1$ bo'lganda, $\int\frac{A}{x-a}dx=A\int\frac{dx}{x-a}=A\ln|x-a|+C.$

2) $m>1$ bo'lganda, $\int\frac{A}{(x-a)^m}dx=A\int(x-a)^{-m}dx=\frac{A}{1-m}\frac{1}{(x-a)^{m-1}}+C.$

3) $k=1$ bo'lganda, $a^2=q-\frac{p^2}{4}, x+\frac{p}{2}=t$ almashtirish olib, kvadrat uchhadni, $x^2+px+q=a^2+t^2$ ko'rinishga keltiramiz, va berilgan kasrning aniqmas integralini topamiz:

$$\int\frac{Bx+D}{x^2+px+q}dx=B\int\frac{tdt}{a^2+t^2}+\frac{(2D-Bp)}{2}\int\frac{dt}{a^2+t^2}=\frac{B}{2}\ln(x^2+px+q)+\frac{2D-Bp}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C$$

4) $k>1$ bo'lganda, $a^2=q-\frac{p^2}{4}, x+\frac{p}{2}=t$ almashtirish olib, olib kvadrat uchhadni, $x^2+px+q=a^2+t^2$ ko'rinishga keltiramiz va berilgan kasrning aniqmas integralini topamiz:

$$\int \frac{Bx+D}{(x^2+px+q)^k}dx = \frac{B}{2}\frac{1}{1-k}\frac{1}{(a^2+t^2)^{k-1}} - \frac{(2D-Bp)}{2}\int \frac{dt}{(a^2+t\ \ )^k}$$

Oxirgi integral esa, quyidagi

$$\int \frac{dt}{(a^2+t^2)^k} = \frac{1}{2(k-1)a^2}\left(\frac{t}{(a^2+t^2)^{k-1}} - (2k-3)\int \frac{dt}{(a^2+t^2)^{k-1}}\right)$$

Rekurrent formula bilan topiladi.

Xulosa qilib aytganda, har qanday haqiqiy koeffitsiyentli haqiqiy o‘zgaruvchili ratsional (kasr ratsional) funksiyaning boshlang‘ich funksiyasi – logarifm, arktangens va ratsional funksiya orqali ifodalanar ekan.

To‘g‘ri ratsional kasrlarni noma’lum koeffitsiyentli (6) ko‘rinishdagi sodda kasrlar yig‘indisi shaklida tasvirlaganda undagi no’mlum koeffitsiyentlarni yuqorida ko‘rsatilgan (*) noma’lum koeffitsiyentlar usulidan foydalanib topishda chiziqli algebraik tenglamalarni yechishga to‘g‘ri keladi, lekin chiziqli tenglamalar sistemasini yechish har doim ham yengil bo‘lavermaydi. Xususiy hollarda noma’lum koeffitsiyentlar usuliga qaraganda qulayroq bo‘lgan, ya’ni noma’lum koeffitsiyentlarni topishda osonroq bo‘lgan usullar mavjud. Masalan, Xevisayd usuli, Gorner sxemasi, differensiallashdan foydalanish usullari shular jumlasidandir.

Xevisayd usuli. Agar $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ to‘g‘ri kasrning maxraji $\frac{P_m(x)}{Q_n(x)}$ ushbu

$$Q_n(x) = a_n(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n), a_n = 1 \qquad (11)$$

Ko‘rinishida tasvirlansa, $\alpha_1, \alpha_2, ..., \alpha_n$ - haqiqiy sonlar bo‘lib , $Q_n(\alpha_i) = 0, (i = \overline{1,n})$,uning (6) shakldagi sodda kasrlarga yoyilmasidagi koeffitsiyentlarning, Xevisayd usulidan foydalanib toppish maqsadga muvofiq bo‘ladi. Bu usulni qisqacha, quyidagi tartibda amalga oshiramiz:

1-qadam. $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ to‘g‘ri kasrning maxraji $Q_n(x)$ ning ifodadagi ko‘paytuvchilar orqali yozish , ya’ni $\frac{P_m(x)}{Q_n(x)} = \frac{P_m(x)}{(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)}$.

2-qadam. $Q_n(x)$ ning $(x-\alpha_i)(i=\overline{1,n})$ ko‘paytiruvchisini vaqtincha yopib, i ning har bir qiymatida yopilmagan ko‘paytiruvchilarda x ni α_i son bilan almashtiramiz. Bu esa, har bir α_i ildiz uchun A_i sonni beradi:

$$A_1 = \frac{P_m(\alpha_i)}{(\alpha_1-\alpha_2)...(\alpha_1-\alpha_n)},$$

$$A_2 = \frac{P_m(\alpha_2)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)...(\alpha_2-\alpha_n)},$$

....

$$A_n = \frac{P_m(\alpha_n)}{(\alpha_n-\alpha_1)(\alpha_n-\alpha_2)...(\alpha_n-\alpha_{n-1})}.$$

3-qadam. $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ ratsional kasrni

$\frac{P_m(x)}{Q_n(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + ... + \frac{A_n}{x-\alpha_n}$ ko‘rinishda yozish.

Misol uchun, bu usul 1-misolga tadbiq qilsak, A, B, C, D, E koeffitsiyentlarni osongina topish mumkin:

1-qadam.

$$\frac{P_4(x)}{Q_5(x)} = \frac{3x^4+3x^3-13x^2+4}{x(x-1)(x+1)(x-2)(x+2)}, \ \alpha_1=0, \ \alpha_2=1, \ \alpha_3=-1, \ \alpha_4=2, \ \alpha_5=-2.$$

2-qadam. $A = \frac{4}{(-1)\cdot 1\cdot(-2)\cdot 2} = 1$

$$B = \frac{-3}{1\cdot 2\cdot(-1)\cdot 3} = \frac{1}{2};$$

$$C = \frac{3-3-13+4}{(-1)\cdot(-2)\cdot(-3)\cdot 1} = \frac{3}{2};$$

$$D = \frac{3\cdot 16+3\cdot 8-13\cdot 4+4}{2\cdot 1\cdot 3\cdot 4} = 1;$$

$$E = \frac{3\cdot 16-3\cdot 8-13\cdot 4+4}{(-2)\cdot(-1)\cdot(-3)\cdot(-4)} = -1;$$

3-qadam. Berilgan $= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} + \frac{E}{x+2} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{3}{2}\cdot\frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x+2}$ to‘g‘ri kasrni,

$$\frac{P_4(x)}{Q_5(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} + \frac{E}{x+2} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{3}{2}\cdot\frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x+2}$$

Ko‘rinishda yozamiz.

Ko‘p hollarda x ning qiymatlarini, masalan, $x = 0, \pm 1, \pm 2, ...$ kabilarni, tanlash yordamida ham noma’lum koeffitsiyentlarni toppish qulay bo‘ladi.

Differensiallashdan foydalanish usuli.

Bu usulni, $\frac{P_m(x)}{Q_n(x)}$ $(m<n)$ to‘g‘ri kasrning $Q_n(x)$ maxraji, haqiqiy karrali ildizlarga ega bo‘lganda qo‘llash qulay bo‘ladi.

4-misol Ushbu

$\frac{(x-1)}{(x+2)^3} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$ tenglamada A, B, C noma’lum koeffitsiyentlarni differensiallashdan foydalanish usuli bo‘yicha topish jarayonini batafsil qarab chiqamiz. A, B, C

Yechilishi: 1-qadam. Dastlab kasrdan qutilamiz:

$$x-1=A(x+2)^2+B(x+2)+C \quad (13)$$

Bu tenglamada, $x=-2$ deb olsak, $C=-3$ bo‘lishini topamiz.

2-qadam. (13) tenglikni ikkala tomonini x ga nisbatan differensiallab,

$$1=2A(x+2)+B \quad (14)$$

tenglikni hosil qilamiz. Bunda $x=-2$deb olsak, $B=1$ bo‘ladi.

3-qadam. Endi (14) tenglikni ikkala tomonini x ga nisbatan differensiallasak, natijada

$$0=2A, A=0$$

bo‘ladi. Demak , $\frac{(x-1)}{(x+2)^3}=\frac{1}{(x+2)^2}-\frac{3}{(x+2)^3}$

Gorner sxemasi.

Har qanday n- darajali $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$ $(a_0\neq 0)$ ko‘phadni $x-a$ ikkihadning darajalari bo‘yicha yoyish mumkin:

$P_n(x)=A_n(x-\alpha)^n+A_{n-1}(x-\alpha)^{n-1}+...+A_2(x-\alpha)^2+A_1(x-\alpha)+A_0$ bunda $A_i(i=\overline{0,n})$ noma’lum koeffitsiyentlar. Gorner sxemasini ketma-ket qo‘llash yordamida, $A_i(i=\overline{0,n})$ noma’lum koeffitsiyentlarni topish jadvalini keltiramiz:

		x^n	x^{n-1}	x^{n-2}	…	x	x^0
		a_n	a_{n-1}	a_{n-2}	…	a_1	a_0
x^0	α	a_n	$\alpha a_n+a_{n-1}=b_{n-1}$	$\alpha b_{n-1}+a_{n-2}=b_{n-2}$	…	$\alpha b_2+a_1=b_1$	$\alpha b_1+a_0=A_0$
x^1	α	a_n	$\alpha a_n+b_{n-1}=c_{n-1}$	$\alpha c_{n-1}+b_{n-2}=c_{n-2}$	…	$\alpha c_2+b_1=A_1$	
x^2	α	a_n	$\alpha a_n+c_{n-1}=d_{n-1}$	$\alpha d_{n-1}+c_{n-2}=d_{n-2}$	…		
x^{n-2}	α	a_n	$\alpha a_n+l_{n-1}=h_{n-1}$	$\alpha h_{n-1}+l_{n-2}=A_{n-2}$			
x^{n-1}	α	a_n	$\alpha a_n+h_{n-1}=A_{n-1}$				
x^n	α	$a_n=A_n$					

$P_n(x)=A_n(x-\alpha)^n+A_{n-1}(x-\alpha)^{n-1}+...+A_2(x-\alpha)^2+A_1(x-\alpha)+A_0$ ko‘phadni $(x-\alpha)^{n+1}$ga bo‘lamiz, $\frac{P_n(x)}{(x-\alpha)^{n+1}}$ natijada to‘g‘ri ratsional kasrni soda kasrlarga yoygan bo‘lamiz:

$$\frac{P_n(x)}{(x-\alpha)^{n+1}}=\frac{A_n}{(x-\alpha)}+\frac{A_{n-1}}{(x-\alpha)^2}+...+\frac{A_2}{(x-\alpha)^{n-2}}+\frac{A_1}{(x-\alpha)^{n-1}}+\frac{A_0}{(x-\alpha)^{n+1}}$$

Ostrogradskiy usuli.

$\frac{P(x)}{Q(x)}$ to‘g‘ri kasrning maxraji karrali kompleks ildizlarga ega bo‘lganda uni integrallashda murakkab hisoblashlarni bajarishga to‘g‘ri keladi. Bunday hollarda ushbu

$$\int \frac{P(x)}{Q(x)}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx \quad (16)$$

Ostrogradskiy formulasidan foydalanish qukay bo‘ladi, bunda $Q_2(x)$ - ildizlari $Q(x)$ ko‘phadning hamma sodda (bir karrali) ildizlaridan iborat bo‘lgan ko‘phad $Q(x) = Q_1(x)\cdot Q_2(x)$ $P_1(x)$ va $P_2(x)$ lar noma’lum koeffitsiyentli ko‘phadlar bo‘lib $\frac{P_1(x)}{Q_1(x)}$ va $\frac{P_2(x)}{Q_2(x)}$ to‘g‘ri kasrlardan iborat.

$P_1(x)$ va $P_2(x)$ ko‘phadlarni topish uchun , ularni noma’lum koeffitsiyentlar yordamida yozib olib so‘ngra ko‘phadlar bo‘lib (16) ning ikkala tomonini differensiallaymiz, natijada (16) tenglikka teng kuchli

$$\frac{P(x)}{Q(x)} = \left(\frac{P_1(x)}{Q_1(x)}\right)' + \frac{P_2(x)}{Q_2(x)}$$

Tenglikka ega bo‘lamiz. Bu tenglikdan noma’lum koeffitsiyentlar usulidan foydalanib, $P_1(x)$ va $P_2(x)$ larning tarkibidagi noma’lum koeffitsiyentlarni topamiz. Ostrogradskiy formulasi intregralning ratsional qismini (integrallamasdan) ajratishga yordam beradi, $\frac{P_1(x)}{Q_1(x)}$ to‘g‘ri kasrni integrallash masalasi unga nisbatan osonroq integrallanadigan $\frac{P_2(x)}{Q_2(x)}$ to‘g‘ri kasrni integrallashga keltiriladi.

FOYDALANILGAN ADABIYOTLAR

1. Azlarov T.A, Mansurov X.T. Matemetik analiz. 1-qism. -T.: “O‘qituvchi” 1994.
2. Azlarov T.A., Mirzaahmedov M.A., Otaqo‘ziyev D.O., Sobirov M.A., To‘laganov S.T. – Matemetikadan qo‘llanma, II qism. T.: “O‘qituvchi”, 1990.
3. Ataxanov K.U., Yerzin V.A., Xodjayev B. – Matematik analizdan misol va masalalar to‘plami, 1-qism, T., 2004.
4. Finney, Weir, Giordano Thomas’CALCULUS. 10-th edition.-Boston, San Francisco, New York, London, Toronto, Sydney, Tokyo. Brisbane Singapore Toronto, 1999, 708 pp.