TO’LQIN TARQALISH TENGLAMASI UCHUN UMUMLASHGAN KOSHI MASALASINI YECHISH

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Annotatsiya: ushbu maqolada to’lqin tarqalish tenglamasi uchun umumlashgan Koshi masalasini yechish qaralgan. Maqolada boshlang’ich shartlar asosida Grin funksiyasidan foydalanib umumiy shakldagi yechimlar topiladi va bir qancha misollar orqali formulalarning amaliyotda qo’llanilishi ko’rsatiladi.

Annotation: This article discusses the solution to the problem of expansion for wave propagation analysis. Using the Green's function under the initial conditions, solutions describing wave propagation are constructed, and through several examples, the practical application of theoretical data and formulas is demonstrated.

Kalit so’zlar: umumlashgan koshi masalasi; Grin funksiyasi; kechikuvchi potensial; oddiy va qo’sh qatlam potensiallari; Lebeg teoremasi.

Keywords: expansion for the Cauchy problem; Green's function; outgoing potential; applicative and multilayer potentials; Lebegue's theorem.

Kirish

Matematik fizika tenglamalaridan biri bo’lgan to’lqin tarqalish tenglamasi ko’plab texnik jarayonlarni modellashtirishda muhim ahamiyatga ega. Xususan, elastik muhitda to’lqin tarqalishi, akustik va elektromagnit to’lqinlarning harakati, suv to’lqinlarining harakati, geofizikada yer osti to’lqinlarining harakati va boshqa jarayonlarni tasvirlashda foydalaniladi. To’lqin tarqalish tenglamasi uchun umumlashgan Koshi masalasini tadqiq qilish esa bu masalalarning boshlang’ich shartlar asosida keyinchalik qanday o’zgarishini ko’rish imkonini beradi. Bu masalalarni tadqiq qilishda fransuz matematiklari Ogisten-Jean Fresnel, Jozef Fure va bundan tashqari S. Sobolev, L. Shvarc, V. Vladimirov kabilarning ilmiy mehnatlari katta ahamiyatga ega.

To’lqin tarqalish tenglamasi uchun quyidagi klassik Koshi masalasi berilgan bo’lsin:

$$\square_a u = u_{tt} - a^2 \Delta u = f(x,t), \tag{1}$$

$$u|_{t=+0}=u_0(x), \qquad u_t|_{t=+0}=u_1(x). \tag{2}$$

Bunda W_a – Dalamber operatori, $f\in C(t\geq 0)$, $u_0\in C^1(\mathbb{R}^m)$ va $u_1\in C(\mathbb{R}^m)$.

Aytaylik, (1)-(2) Koshi masalasining klassik $u(x,t)$ yechimi mavjud bo'lsin, ya'ni $u\in C^2(t>0)\cap C^1(t\geq 0)$ sinfga tegishli bo'lgan, $t>0$ qiymatlarda (1) tenglamani, $t\to+0$ da esa (2) boshlang'ich shartlarni qanoatlantiruvchi funksiya aniqlangan.

Endi u va f funksiyalarni $t<0$ yarim o'qda

$$\tilde{u}=\begin{cases}u, & t\geq 0,\\ 0, & t<0,\end{cases} \qquad \tilde{f}=\begin{cases}f, & t\geq 0,\\ 0, & t<0\end{cases}$$

nol qilib davom ettiramiz. Keyin $\tilde{u}(x,t)$ funksiyaning $\mathbb{R}^{m+1}$ da quyidagi to'lqin tarqalishi tenglamasining yechimi ekanligini ko'rsatamiz:

$$W_a\tilde{u}=\tilde{f}(x,t)+u_0(x)\cdot\delta'(t)+u_1(x)\cdot\delta(t) \tag{3}$$

Haqiqatan ham, $\forall\varphi(x,t)\in D(\mathbb{R}^{m+1})$ uchun quyidagi tengliklarga ega bo'lamiz:

$$\langle W_a\tilde{u},\varphi\rangle=\langle\tilde{u},W_a\varphi\rangle=\int_0^\infty dt\int_{\mathbb{R}^m}u\,W_a\varphi dx=$$

$$=\lim_{\varepsilon\to+0}\int_\varepsilon^{+\infty}\int_{\mathbb{R}^m}u\left(\frac{\partial^2\varphi}{\partial t^2}-a^2\Delta\varphi\right)dxdt=\lim_{\varepsilon\to+0}\left[\int_\varepsilon^{+\infty}\int_{\mathbb{R}^m}\left(\frac{\partial^2 u}{\partial t^2}-a^2\Delta u\right)\varphi dxdt-\right.$$

$$\left.-\int_{\mathbb{R}^m}\frac{\partial\varphi(x,\varepsilon)}{\partial t}u(x,\varepsilon)dx+\int_{\mathbb{R}^m}\varphi(x,\varepsilon)\frac{\partial u(x,\varepsilon)}{\partial t}dx\right]=$$

$$=\int_0^{+\infty}\int_{\mathbb{R}^m}f\varphi dxdt-\int_{\mathbb{R}^m}\frac{\partial\varphi(x,0)}{\partial t}u(x,0)dx+\int_{\mathbb{R}^m}\varphi(x,0)\frac{\partial u(x,0)}{\partial t}dx=$$

$$=\int_{\mathbb{R}^{m+1}}\tilde{f}\varphi\,dx\,dt-\int_{\mathbb{R}^m}u_0(x)\frac{\partial\varphi(x,0)}{\partial t}dx+\int_{\mathbb{R}^m}u_1(x)\varphi(x,0)\,dx=$$

$$=\left\langle\tilde{f}(x,t)+u_0(x)\cdot\delta'(t)+u_1(x)\cdot\delta(t),\varphi(t,x)\right\rangle.$$

(3) tenglikda $\tilde{u}(x,t)$ funksiyasi uchun boshlang'ich shartlar bo'lgan $u_0(x)$ va $u_1(x)$ funksiyalari $t=0$ vaqt momentida tez ta'sir qiluvchi $u_0(x)\cdot\delta'(t)+u_1(x)\cdot\delta(t)$ tashqi ta'sir kuchiga aylanadi, ya'ni $u_0(x)$ boshlang'ich ta'sirga $u_0(x)\cdot\delta'(t)$ qo'sh qatlam, $u_1(x)$ ga esa $u_1(x)\cdot\delta(t)$ oddiy qatlam mos keladi.

Shu bilan birga (1)-(2) klassik Koshi masalasining yechimlari $t<0$ qiymatlarda nolga aylanadigan yechimlarga ega bo'ladigan (3) umumlashgan Koshi masalasining yechimlari orasida bo'ladi. U holda tashqi ta'sir kuchini umumlashgan funksiya deb hisoblab

$$F(x,t)=\tilde{f}(x,t)+u_0(x)\cdot\delta'(t)+u_1(x)\cdot\delta(t)$$

belgilash kiritamiz.

1-ta'rif [1, 2, 3]. Quyidagi ko'rinishdagi bir jinsli bo'lmagan to'lqin tenglamasini

$$W_a u=u_{tt}-a^2\Delta u=F(x,t), \qquad F\in D'(\mathbb{R}^{m+1}) \tag{4}$$

qanoatlantiruvchi va $t<0$ oraliqda nolga aylanuvchi $u(x,t)\in D'(\mathbb{R}^{m+1})$ umumlashgan funksiyani topish masalasiga umumlashgan Koshi masalasi deb ataladi.

(4) tenglama $D'(\mathbb{R}^{m+1})$ fazosida quyidagi tenglikka teng kuchli, ya'ni $\forall\varphi\in D(\mathbb{R}^{m+1})$ uchun

$$\langle u, W_a\varphi\rangle=\langle F,\varphi\rangle. \tag{5}$$

(4) tenglamadan F funksiyasi $t<0$ oraliqda nolga aylanuvchi umumlashgan Koshi masalasi yechimga ega bo'lishining zaruriy sharti bo'ladi. Keyingi tasdiqlar ushbu shartning yetarli ekanligini ham oydinlashtiradi.

Eslatma [1, 2, 3]. Biz (3) ni keltirib chiqarishda, $u\in C^2(t>0)\cap C^1(t\geq 0)$ sinfga tegishli va $t<0$ oraliqda nolga aylanuvchi, Dalamber operatorining ta'siri $W_a u\in C(t\geq 0)$ sinfga tegishli bo'lgan ixtiyoriy $u(x,t)$ funksiya uchun bajariluvchi quyidagi tenglikni ko'rsatdik:

$$W_a u=\{W_a u(x,t)\}+u(x,0+)\cdot\delta'(t)+u_t'(x,0+)\cdot\delta(t). \tag{6}$$

1-teorema [1, 2, 3]. $F(x,t)\in D'(\mathbb{R}^{m+1})$ umumlashgan funksiyasi $t<0$ oraliqda $F(x,t)=0$, $E_m(x,t)-$Dalamber operatorining Grin funksiyasi bo'lsin. U holda (4) umumlashgan Koshi masalasining yechimi mavjud, yagona va quyidagi kechikuvchi potensial ko'rinishida aniqlanadi:

$$u=E_m*F. \tag{7}$$

Bu yechim D' fazosida F funksiyalarga bog'liq bo'lmaydi.

1-natija [1, 2, 3]. $u\in C^2(t>0)\cap C^1(t\geq 0)$ sinfga tegishli va $t<0$ oraliqda nolga aylanuvchi Dalamber operatorining ta'siri esa $W_a u\in C(t\geq 0)$ sinfga tegishli bo'lgan ixtiyoriy $u(x,t)$ funksiyasi quyidagi ko'rinishda ifodalanadi:

$$u(x,t)=V_m(x,t)+V_m^{(0)}(x,t)+V_m^{(1)}(x,t). \tag{8}$$

Bunda $V_m(x,t)$ zichligi $\{W_a u\}$ ga teng bo'lgan kechikuvchi potensial, $V_m^{(0)}(x,t)$ va $V_m^{(1)}(x,t)$ lar esa zichliklari mos ravishda $u_t'(x,+0)$ va $u(x,+0)$ ga teng bo'lgan oddiy va qo'sh qatlam potensiallari, ya'ni

$$V_m(x,t) = E_m(x,t) * \tilde{f}(x,t),$$

$$V_m^{(0)}(x,t) = E_m(x,t) * [u_1(x)\cdot\delta(t)] = E_m(x,t) * u_1(x),$$

$$V_m^{(1)}(x,t) = E_m(x,t) * [u_0(x)\cdot\delta'(t)] = \frac{\partial}{\partial t}\big(E_m(x,t) * u_0(x)\big).$$

Bunda $u(x,t)$ funksiyasi (6) tenglamani qanoatlantirgani uchun 1-teorema ga ko'ra bu funksiya kechikuvchi potensiallar uchligining (8) ko'rinishdagi yig'indisi sifatida aniqlanadi.

1-misol. Ushbu berilganlar bo'yicha umumlashgan Koshi masalasini yechishni qaraymiz:

$$f(x,t) = \theta(t)\sin t\cdot\delta(x-x_0), \qquad u_0(x) = 0, \qquad u_1(x) = x\delta'(x)$$

ya'ni

$$\begin{cases} u_{tt} - u_{xx} = \theta(t)\sin t\cdot\delta(x-x_0), \\ u|_{t=+0} = 0, \qquad u_t|_{t=+0} = x\delta'(x). \end{cases}$$

Yechish. Bunda $V^{(0)}(x,t)$, $V^{(1)}(x,t)$ oddiy va qo'sh qatlam potensiallari mos ravishda quyidagicha aniqlanadi:

$$V^{(0)}(x,t) = E_1(x,t) * [\delta'(t)\cdot u_0(x)] = 0,$$

$$V^{(1)}(x,t) = E_1(x,t) * [\delta(t)\cdot x\delta'(x)] \overset{(x\delta'(x) = -\delta(x))}{=} -E_1(x,t) * [\delta(t)\cdot\delta(x)] =$$

$$= -E_1(x,t) = -\frac{1}{2a}\theta(at-|x|).$$

U holda bundan kechikuvchi potensial quyidagicha aniqlanadi:

$$V(x,t) = E_1(x,t) * [\theta^2(t)\sin t\cdot\delta(x-x_0)] \overset{(3^0)}{=}$$

$$\overset{(3^0)}{=} \frac{1}{2a}\theta(at-|x-x_0|)\int_0^{t-\frac{|x-x_0|}{a}} \theta^2(\tau)\cdot\sin\tau d\tau =$$

$$= \frac{1}{2a}\theta(at-|x-x_0|)\left[1-\cos\left(t-\frac{|x-x_0|}{a}\right)\right].$$

Shunday qilib, nerilgan boshlang'ich shartlarda to'lqin tarqalishi tenglamasi uchun umumlashgan Koshi masalasining yechimi quyidagi ko'rinishda bo'ladi:

$$u(x,t)=\frac{1}{2a}\theta(at-|x-x_0|)\left[1-\cos\left(t-\frac{|x-x_0|}{a}\right)\right]-\frac{1}{2a}\theta(at-|x|).$$

2-misol [4]. Quyidagi Koshi masalasining yechimini toping:

1. $$\begin{cases} u_{tt}=u_{xx}+e^x, \ t>0, \ x\in\mathbb{R}^1, \\ u|_{t=+0}=\sin x, \quad u_t|_{t=+0}=x+\cos x. \end{cases}$$

Yechish. Bu misolda umumlashgan Koshi masalasi:

$$\tilde{u}_{tt}-\tilde{u}_{xx}=\theta(t)e^x+\delta'(t)\cdot\sin x+\delta(t)\cdot(x+\cos x), \qquad \mathrm{E}_1(x,t)=\frac{1}{2a}\theta(at-|x|),$$

1) $$\mathrm{E}_1(x,t)*[\delta(t)\cdot(x+\cos x)]=\mathrm{E}_1(x,t)*(x+\cos x)=$$

$$=\int_{\mathbb{R}^1}\mathrm{E}_1(x-\xi,t)\cdot(\xi+\cos\xi)d\xi=\frac{1}{2}\int_{\mathbb{R}^1}\theta(t-|x-\xi|)\cdot(\xi+\cos\xi)d\xi=$$

$$=\frac{1}{2}\int_{x-t}^{x+t}(\xi+\cos\xi)d\xi=\frac{1}{2}\left(\frac{\xi^2}{2}+\sin\xi\right)\Bigg|_{\xi=x-t}^{x+t}=xt+\sin t\cos x;$$

2) $$\mathrm{E}_1(x,t)*[\delta'(t)\cdot\sin x]=\frac{\partial\mathrm{E}_1(x,t)}{\partial t}*\sin x=$$

$$=\frac{1}{2}\frac{\partial}{\partial t}(\theta(t-|x|))*\sin x=\frac{1}{2}\delta(t-|x|)*\sin x=$$

$$=\frac{1}{2}[\theta(t)\delta(t+x)+\theta(t)\delta(t-x)]*\sin x=$$

$$=\frac{1}{2}\theta(t)\delta(t+x)*\sin x+\frac{1}{2}\theta(t)\delta(t-x)*\sin x=$$

$$=\frac{\theta(t)}{2}\left(\int\sin\xi\cdot\delta(\xi-(x+t))d\xi+\int\sin\xi\cdot\delta(\xi-(x-t))d\xi\right)=$$

$$=\frac{\theta(t)}{2}[\sin(x+t)+\sin(x-t)]=\theta(t)\sin x\cos t;$$

3) $$\mathrm{E}_1(x,t)*[\theta(t)\cdot e^x]=\int\int\theta(\tau)e^{\xi}\cdot\mathrm{E}_1(x-\xi,t-\tau)d\xi d\tau=$$

$$=\frac{1}{2}\int\int\theta(\tau)e^{\xi}\cdot\theta(t-\tau-|x-\xi|)d\xi d\tau=$$

$$\begin{pmatrix} t-\tau-|x-\xi|\geq 0 \ \Rightarrow \ |x-\xi|\leq t-\tau \ \Rightarrow \\ \Rightarrow \ 0<\tau<t \ \Rightarrow \ -(t-\tau)\leq\xi-x\leq t-\tau \ \Rightarrow \\ \Rightarrow \ x-(t-\tau)\leq\xi\leq x+(t-\tau). \end{pmatrix}$$

$$=\frac{1}{2}\int_0^t d\tau\int_{x-(t-\tau)}^{x+(t-\tau)}e^{\xi}d\xi=\frac{1}{2}\int_0^t\left(e^{x+(t-\tau)}-e^{x-(t-\tau)}\right)d\tau=$$

$$=\frac{1}{2}\left[-e^{x}+e^{x+t}-e^{x}+e^{x-t}\right]=e^{x}\left(\frac{e^{t}-e^{-t}}{2}-1\right)=-e^{x}(1-\operatorname{ch}t).$$

Demak, yakuniy javob quyidagi ko'rinishda bo'ladi:

$$u(x,t)=xt+\sin(x+t)-e^{x}(1-\operatorname{ch}t).$$

3-msol [4]. $t\ge 0,\quad x\in\mathbb{R}^1$ o'zgaruvchilarida quyidagi tashqi ta'sir kuch manbayiga ega bo'lgan to'lqin tarqalish tenglamasi uchun umumlashgan koshi masalasini yeching:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u(t,x)=\frac{\delta(x)\theta(t)}{\sqrt{t}},\qquad a=1.$$

Yechish. Ushbu shartlardagi to'lqin tarqalish tenglamasining Grin funksiyasi

$$\begin{cases} \mathrm{E}_1(x,t)=\dfrac{1}{2a}\theta(at-|x|)\overset{(a=1)}{=}\dfrac{1}{2}\theta(t-|x|), \qquad \operatorname{supp}\mathrm{E}_1=\{t\ge |x|\},\\ \mathrm{W}_a\mathrm{E}_1(x,t)=\delta(x,t)\end{cases}$$

tashqi kuch funksiyasi va uning tashuvchisi esa

$$F(x,t)=\frac{\delta(x)\theta(t)}{\sqrt{t}},\qquad \operatorname{supp}F=\{t\ge 0,\ x=0\}.$$

Yuqorida keltirilgan natijalar bo'yicha

$$u_{tt}-u_{xx}=\frac{\delta(x)\theta(t)}{\sqrt{t}}$$

Koshi masalasining yechimi quyidagi formula bilan aniqlanadi:

$$u(x,t)=V(x,t)=\mathrm{E}_1(x,t)*F(t,x)=\frac{1}{2}\theta(t-|x|)*\frac{\delta(x)\theta(t)}{\sqrt{t}}.$$

Bunda

$$\operatorname{supp}\left(\mathrm{E}_1*F\right)\subset\overline{\{t\ge 0,\ x=0\}\cup\{t\ge |x|\}}=\{t\ge |x|\}.$$

Endi berilgan masalani yechish uchun to'lqin tarqalish operatorining Grin funksiyasi va tashqi kuch funksiyasining umumlashgan o'ramasini ta'rif bo'yicha hisoblaymiz, ya'ni $\forall\varphi(t,x)\in S(\mathbb{R}^2)$ asosiy va ixtiyoriy $1-$ kesim $\eta_1(t,x)\in D(\mathbb{R}^2)$ funksiyalari uchun

$$\left\langle \mathrm{E}_1*F,\varphi\right\rangle=\lim_{R\to+\infty}\left\langle F(t,x),\ \eta_1\left(\frac{t}{R},\frac{x}{R}\right)\left\langle \mathrm{E}_1(\tau,y),\varphi(t+\tau,\,x+y)\right\rangle\right\rangle=$$

$$=\lim_{R\to+\infty}\int_0^{+\infty}\frac{dt}{\sqrt{t}}\,\eta_1\left(\frac{t}{R},0\right)\iint_{\tau\ge |y|}\frac{1}{2}\varphi(t+\tau,\,y)\,d\tau\,dy.$$

Keyin limitga o'tish amalini integral belgisi ostiga kiritish uchun chegaralangan yaqinlashuvchilik haqidagi Lebeg teoremasi shartlarini tekshirib ko'ramiz:

Ushbu shartlarni keltirib chiqarish uchun $\varphi(t,x)$ ning asosiy funksiya ekanligidan foydalanamiz, ya'ni

$$\exists C>0: \quad \forall t\geq 0, \ \ \forall z\in \mathbb{R}^1 \to |\varphi(t,x)|\leq \frac{C}{(1+t^2)^3}.$$

Bundan $\forall t\geq 0, \ \ \forall \tau\geq |y|$ uchun

$$|\varphi(t+\tau,x)|\leq \frac{C}{[1+(t+\tau)^2]^3}\leq \frac{C}{(1+t^2)(1+\tau^2)^2}.$$

Shu bilan birga kesim funksiya uchun quyidagi tasdiqlar o'rinli bo'ladi. Yetarli darajada katta R va $\forall\, t\geq 0$, $\forall x\in \mathbb{R}^1$ uchun

$$\lim_{R\to +\infty} \eta_1\left(\frac{t}{R},\frac{x}{R}\right)=1$$

va $1-$kesim $\eta_1(t,x)\in D(\mathbb{R}^2)$ uzluksiz funksiya sifatida chegaralangan, ya'ni

$$\exists M>0: \quad \forall t\geq 0, \quad \forall x\in \mathbb{R}^1 \ \to \ |\eta_1(t,x)|\leq M\,.$$

Bundan integral belgisi ostidagi funksiya absolyut integrallanuvchi bo'ladi, ya'ni

$$\left| \frac{1}{\sqrt{t}}\,\eta_1\left(\frac{t}{R},0\right)\varphi(t+\tau,\,y)\right| \leq \frac{M}{\sqrt{t}\,(1+t^2)}\cdot\frac{C}{(1+\tau^2)^2}\in L_1\left(\begin{matrix} t>0\\ \tau\geq |y|\end{matrix}\right).$$

Demak, ushbu shartlarda chegaralangan yaqinlashuvchilik haqidagi Lebeg teoremasi shratlari o'rinli bo'lib, biz limitga o'tish amalini integral belgisi ostiga kiritishimiz mumkin bo'ladi. Keyingi hisoblashlar R ga bog'liq bo'lmagan o'rama ifodasi bilan bog'liq bo'ladi, ya'ni

$$\langle \mathrm{E}_1 * F,\varphi\rangle = \int_0^{+\infty}\frac{dt}{\sqrt{t}}\iint_{\tau\geq |y|}\frac{1}{2}\,\varphi(t+\tau,y)d\tau\, dy \overset{(\xi=t+\tau)}{=}$$

$$=\int_0^{+\infty}\frac{dt}{\sqrt{t}}\iint_{\xi\geq t+|y|}\frac{1}{2}\varphi(\xi,y)\,d\xi dy = \iint_{\xi\geq |y|}\frac{1}{2}\,\varphi(\xi,y)\,d\xi dy\int_0^{\xi-|y|}\frac{dt}{\sqrt{t}} =$$

$$=\iint_{\xi\geq |y|}\sqrt{\xi-|y|}\,\varphi(\xi,y)\,d\xi dy = \left\langle \sqrt{t-|x|}\,\theta(t-|x|),\,\varphi(t,x)\right\rangle.$$

Shunday qilib

$$u(x,t)=\frac{1}{2}\theta(t-|x|)*\frac{\delta(x)\theta(t)}{\sqrt{t}}=\sqrt{t-|x|}\,\theta(t-|x|).$$

Xulosa

Ushbu maqolada to'lqin tarqalishi tenglamasi uchun Koshi masalasi uchun umumlashgan yechimlarni topish masalasi qaraldi. Maqolada beilgan nazariy ma'lumotlar asosida bir qancha misollar ko'rib chiqildi va ular matematik fizika masalalarini umumlashgan funksiyalar nazariyasi yordamida yechishga asos bo'lib xizmat qiladi.

Foydalanilgan adabiyotlar ro'yxati

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