CONNECTION BETWEEN CLASSICAL AND MODERN METHODS FOR SOLVING FIRST AND SECOND-ORDER DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

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Abstract: This article examines the relationship between classical and modern methods for solving first and second-order differential equations and the differences in their solutions. Classical methods are based on analytical solutions, while modern methods utilize numerical techniques to solve complex equations. The connection between classical and modern methods is explained, and their role in mathematical modeling is analyzed. Modern methods, with the aid of computers, provide significant opportunities for modeling large and complex systems.

Keywords: Method of undetermined coefficients, solving homogeneous and nonhomogeneous equations, finite element method, analytical solution, boundary value methods.

Introduction

Differential equations are used to model numerous physical, technological, and engineering processes. They describe the changes in objects or systems. While first and second-order differential equations have traditionally been solved using classical methods through analytical solutions, with the development of modern technologies and complex systems, modern numerical methods have become widely applied. This article reviews classical and modern methods, analyzing their differences and interconnections.

Methodological Analysis

1. First-Order Differential Equations

Classical Methods: Classical methods are frequently used to describe various physical, chemical, and technological processes. These methods are widely applied for analytically solving first-order differential equations. Among them, the method of separation of variables and the integrating factor method are the most commonly used.

- **Separation of Variables Method:** The separation of variables method is one of the most fundamental and straightforward techniques for solving first-order differential equations. This method relies on representing variables in a separable form and is applied to equations of the following form:

$$\frac{dy}{dx} = f(x)g(y)$$

1.

Here, f(x) depends only on x, and g(y) depends only on y. The essence of the method is to separate the variables into two parts:

$$\frac{1}{g(y)}dy = f(x)dx$$

2.

Then, each part is integrated separately:

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

3.

The solution is obtained by integrating the separated variables. This method is applicable only to equations where variables can be separated but is highly effective for many simple systems.

Practical Example: Let's solve the following simple equation:

$$\frac{dy}{dx} = 2xy$$

Separate the variables :

$$\frac{1}{y}dy = 2xdx$$

Integration Result:

$$\int \frac{1}{y}dy = \int 2xdx$$

$$ln \mid y \mid = x^2 + C$$

Express the solution in terms of y:

$$y = Ce^{x^2}$$

This is the general solution obtained using the separation of variables method.

Integrating Factor Method: The integrating factor method is also used to solve first-order differential equations but requires additional steps. The simple integrating factor method is applied to linear equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P(x) and Q(x) - are functions of x. To solve this equation using the integrating factor method, the following steps are taken:

Multiply the equation by the integrating factor μ(x):

$$\mu(x) = e^{\int P(x)dx} \qquad 4.$$

Multiply each term of the equation by the integrating factor μ(x):

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \qquad 5.$$

The equation now takes the form:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \qquad 6.$$

Now integrate the entire equation:

$$\mu(x)y = \int \mu(x)Q(x)dx + C \qquad 7.$$

Here, C is an arbitrary constant. Then, solve for y:

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right) \qquad 8.$$

The separation of variables and integrating factor methods are classical approaches for solving first-order differential equations and are widely used in modeling simple physical and mathematical processes. While the separation of variables method directly integrates separated variables, the integrating factor method is applied to more complex equations. Understanding these methods provides a foundation for modern numerical methods and is useful for solving many engineering problems.

2. Second-Order Differential Equations

Method of Undetermined Coefficients: Second-order differential equations are often used to solve physical problems, particularly in modeling vibrations, heat conduction, and electromagnetic processes. One of the classical methods is the method of undetermined coefficients, commonly used to solve second-order equations. The equation is written in the following form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad 9.$$

This equation is transformed into a characteristic equation:

$$ar^2 + br + c = 0 \qquad 10.$$

By solving the characteristic equation, the general solution is obtained. If the discriminant is positive, it has two real roots, and the solution is written as:

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x} \qquad 11.$$

Here r_1 and r_2 are the roots of the characteristic equation, C_1 and C_2 – are arbitrary constants.

Example: Consider the following equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

The characteristic equation is:

$$r^2 - 5r + 6 = 0$$

Thus, the general solution is:

$$y(x) = C_1e^{2x} + C_2e^{3x}$$

Two Identical Real Roots: If the discriminant is zero, the characteristic equation has two identical real roots. In this case, the general solution is written as:

$$y(x) = (C_1 + C_2 x)e^{rx} \qquad 12.$$

Two Complex Roots: If the discriminant is negative, the equation has two complex roots of the form:

$$r = \alpha + i\beta$$

In this case, the general solution is expressed in terms of trigonometric functions:

$$y(x) = e^{\alpha x}(C_1 cos(\beta x) + C_2 sin(\beta x)) \qquad 13.$$

Solving Homogeneous and Nonhomogeneous Equations:

Homogeneous Equations: These are equations where the right-hand side is zero, with the general form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad 14.$$

Homogeneous equations are solved using classical methods, such as the method of undetermined coefficients described above.

Nonhomogeneous Equations: These are equations with a non-zero right-hand side, of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

15.

Here, f(x) is a given function on the right-hand side. The general solution for nonhomogeneous equations consists of two parts:

- The general solution to the homogeneous equation.
- A particular solution (for the nonhomogeneous part).
- To find the particular solution, various methods, such as the method of variation of parameters or the method of undetermined coefficients, are used. For example, if the right-hand side contains a sine or cosine function, the solution can be found using a particular solution corresponding to that function.
- Second-order differential equations are widely used in engineering, physics, and other scientific processes. The method of undetermined coefficients is highly effective for solving simple differential equations and is used to find general solutions for various cases. Unlike homogeneous equations, nonhomogeneous equations require particular solutions for the nonhomogeneous part. Understanding these methods is of great importance in modern mathematical modeling and is applied to solve many scientific and technical problems.

3. Modern Methods

Classical methods for solving second-order differential equations often provide exact analytical solutions, but in the real world, many problems are too complex, making it difficult or impossible to find analytical solutions. Therefore, modern numerical methods are widely used to solve these equations. Below, two such methods—finite element method and boundary value methods—are discussed in detail.

a) Finite Element Method (FEM)

The Finite Element Method (FEM) is a modern mathematical approach used to model complex geometries and loading conditions. It is widely applied in engineering, physics, and other technical fields, particularly in structural analysis, fluid mechanics, electromagnetic fields, and many other areas.

The finite element method involves dividing the domain into smaller, simpler shapes (elements) and solving the equation for each element. The solutions for these elements are combined to obtain the solution for the entire domain. This approach includes the following steps:

- **Discretization:** The geometry is divided into small elements (e.g., triangles or quadrilaterals).
- **Element Equations:** Equations are formulated for each element, considering their boundary conditions.
- **Solution:** The equations for each element are solved, and the results are combined to obtain the solution for the entire domain.

The finite element method is highly effective for second-order differential equations, especially in complex systems where exact solutions are difficult to obtain.

Practical Applications:

- **Structural Analysis:** FEM is used to analyze the mechanical state of structures, such as stress and deformation.
- **Fluid Mechanics:** FEM is widely applied in modeling fluid flow (based on Navier-Stokes equations).
- **Electronics and Electromagnetism:** Electric fields and electromagnetic wave propagation are analyzed using FEM. [11, pp. 39-41]

b) Boundary Value Methods (BVP)

Boundary Value Methods (BVP) are numerical methods based on boundary conditions for solving differential equations and are commonly used in engineering and science. In this method, the solution to the equation must be found over an entire interval, where boundary conditions are specified. These conditions often represent physical or technical processes. Boundary value methods are significant numerical techniques for differential equations, as many engineering problems (e.g., heat conduction, fluid flow, elasticity) are boundary value problems.

Examples:

- **Heat Conduction:** For instance, if a rod is held at a given temperature at one end and heat flow is calculated at the other, FEM and BVP methods are used to solve the heat conduction differential equation.
- **Electromagnetic Fields:** Modeling the propagation of electromagnetic waves relies on numerical solutions based on boundary conditions.

Boundary value methods involve the following steps:

- **Discretization:** Similar to the finite element method, the domain or system is divided into smaller parts.
- **Solving Differential Equations:** The equations for each part are solved using numerical methods.
- **Applying Boundary Conditions:** Solutions for all parts are combined to obtain the solution for the entire system.

Practical Applications:

- **Fluid Motion:** Modeling the flow of liquids and gases.
- **Heat Problems:** Addressing energy conservation and heat conduction issues.
- **Mechanical Structures:** Calculating stress and deformation in machine parts under load.

Conclusion

Methods for solving first and second-order differential equations are of great importance in modern technology and science. Classical methods, such as separation of variables and integrating factor methods, are effective for analytically solving simple differential equations, but with the advancement of modern technologies, numerical methods have been increasingly applied to solve more complex processes. Modern methods, such as the finite element method and boundary value methods, are widely used in modeling complex geometric shapes, physical processes, and precise technological systems. They enable accurate modeling and analysis of real-world problems in fields such as heat conduction, fluid mechanics, and mechanical structures. Thus, classical and modern methods complement each other, providing extensive opportunities for working with differential equations. Applying these methods to solve complex technological and scientific problems enhances efficiency and ensures the proper functioning of systems. In the future, deeper application of these methods will enable the creation of innovative approaches for working with complex systems.

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