SOLVING DEFINITE INTEGRALS USING NUMERICAL METHOD

Shaxobiddin Sobirovich Kuziev

Kokand University, Kokand, Uzbekistan

shaxobiddin.qoziyev.89@gmail.com

Abstract. In this work, an algorithm for constructing the effective formula for numerical computation of definite integrals in $L_2^{(m)}(0,1)$ space is given. Using this algorithm, a derivative quadrature formula is constructed using the first and second order derivatives of the function for equally distributed nodal points on the section [0,1]. For this, the form of the norm of the optimal error functional is found. Finding the conditional extremum of the function was used to minimize the norm of the error function. A system of linear algebraic equations for optimal coefficients is obtained. It is proved that the solution of the system of equations exists and is unique using the Wandermonde determinant. The formula constructed in this work gives good results if the values of the derivatives of the function at the nodes are given.

Keywords. Sobolev space, optimal error function, Lagrange multipliers, Wandermonde determinant, derivative optimal quadrature formula.

INTRODUCTION

As a result of numerous scientific and practical researches carried out on a global scale, modeling of synthesized holograms, mechanics of liquids and gases leads to the construction of optimal quadrature formulas. Usually, the use of simple interpolation quadrature formulas in solving such problems requires large-scale computing. Therefore, creating optimal numerical solution algorithms that allow calculating the solutions of typical problems in mathematics with sufficient accuracy and developing ways to use modern computing tools for this purpose, as well as constructing optimal quadrature formulas derived in certain Gilbert and Banach spaces and evaluating their errors are important tasks of computational mathematics. is one of the directions. Nowadays, the construction of derivative optimal quadrature formulas in the approximate calculation of exact integrals is of great importance. In particular, using the values of the functions up to the second-order derivative at the nodes in certain Gilbert and Banach spaces, the construction of derivative optimal quadrature formulas and the estimation of their errors are widely used. Professor Kh.M. Shadimetov and his students S.L. Sobolev's research continues, and new results are obtained on the construction of formulas in $\overset{\circ}{L}{}_2^{(m)}(0,1)$, $L_2^{(m)}(R)$, $K_2^{(m)}(P)$ spaces [1]. We consider this formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) + \frac{h^2}{12}(\varphi'(0)-\varphi'(1)) + \sum_{\beta=0}^{N} C_1[\beta]\varphi''(h\beta) \qquad (1)$$

this difference (1) is called the error of the quadrature formula

$$(1_N,\varphi) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) - \frac{h^2}{12}(\varphi'(0)-\varphi'(1)) - \sum_{\beta=0}^{N} C_1[\beta]\varphi''(h\beta)$$

the error functional corresponding to this difference is

$$1_N(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-h\beta) + \frac{h^2}{12}(\delta'(x)-\delta'(x-1)) - \sum_{\beta=0}^{N} C_1[\beta]\delta''(x-h\beta), \qquad (2)$$

where, $\delta(x)$ is the Dirac's delta-function, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $C_1[\beta]$ are the unknown coefficients of the quadrature formula (1), $f \in L_2^{(m)}(0,1)$, $L_2^{(m)}(0,1)$ is the space of functions which are square integrable with m-th generalized derivative.

The problem of constructing a derived quadrature formula of the form (1) in the $L_2^{(m)}(0,1)$ space is given [2].

ANALYTICAL EXPRESSIONS FOR THE COEFFICIENTS OF THE QUADRATIC FORMULA

To find the coefficients of the derivative optimal quadrature formula, we have the following system of equations [22]

$$\sum_{\gamma=0}^{N} C_1[\gamma]\frac{|h\beta-h\gamma|^{2m-5}}{2(2m-5)!}+P_{m-3}(h\beta)=f_m(h\beta),\qquad \beta=\overline{0,N}, \tag{3}$$

$$f_m(h\beta)=\sum_{i=0}^{2m-5}\frac{(h\beta)^{2m-5-i}}{(2m-5-i)!}\left[-\frac{B_{i+3}h^{i+3}}{(i+3)!}+\sum_{j=1}^{i}\frac{(-1)^i B_{i+3-j}h^{i+3-j}}{2j!(i+3-j)!}\right]+\frac{B_{2m-2}h^{2m-2}}{(2m-2)!} \tag{4}$$

$$\sum_{\beta=0}^{N} C_1[\beta](h\beta)^{\alpha}=-\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}h^{\alpha+3-j}}{j!(\alpha+3-j)!},\qquad \alpha=\overline{0,m-3}. \tag{5}$$

To solve a system of linear algebraic equations (3) - (5) with $m\geq 4$ in $L_2^{(m)}(0,1)$ space, we use the approach based on $D_{m-2}[\beta]$ discrete analog of d^{2m-2}/dx^{2m-2} differential operator. For this, $\beta<0$ and $\beta>N$ are $C_1[\beta]=0$, and we rewrite equation (3) in the convolutional form

$$G_{m-2}(h\beta)*C_1[\beta]+P_{m-3}(h\beta)=F_m(h\beta),\qquad \beta=0,1,...,N \tag{6}$$

and enter the following definitions:

$$v(h\beta)=G_{m-2}(h\beta)*C_1[\beta], \tag{7}$$

$$u(h\beta)=v(h\beta)+P_{m-3}(h\beta) \tag{8}$$

$D_{m-2}[\beta]$ discrete analogue of d^{2m-4}/dx^{2m-4} differential operator satisfying $hD_{m-2}(h\beta)*G_{m-2}(h\beta)=\delta(h\beta)$ equality was constructed and its properties were studied.

$$C_1[\beta]=hD_{m-2}(h\beta)*u(h\beta). \tag{9}$$

To calculate the convolution of (7), we need to find the representation of the function $u(h\beta)$ for all integer values of β. If $h\beta\in[0,1]$, $u(h\beta)=f_m(h\beta)$. For $\beta<0$ and $\beta>N$ we find the representation of $u(h\beta)$.

For $\beta<0$

$$v(h\beta)=C_1[\beta]*G_{m-2}(h\beta)=C_1[\beta]*\frac{|h\beta|^{2m-5}}{2(2m-5)!}=\sum_{\gamma=0}^{N}C_1[\gamma]\frac{|(h\beta)-(h\gamma)|^{2m-5}}{2(2m-5)!}=$$

$$=\sum_{i=0}^{m-3}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}-\sum_{i=m-2}^{2m-5}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i.$$

for $\beta>N$

$$v(h\beta)=-\sum_{i=0}^{m-3}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}+\sum_{i=m-2}^{2m-5}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i.$$

We introduce the following definitions

$$R_{2m-5}(h\beta)=\sum_{i=0}^{m-3}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!},$$

$$Q_{m-3}(h\beta)=\sum_{i=m-2}^{2m-5}\frac{(h\beta)^{2m-5-i}(-1)^i}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^i \tag{10}$$

after these designations

$$v(h\beta)=\begin{cases}R_{2m-5}(h\beta)-Q_{m-3}(h\beta), & \beta<0,\\ -R_{2m-5}(h\beta)+Q_{m-3}(h\beta), & \beta>N.\end{cases} \tag{11}$$

So,

$$u(h\beta)=\begin{cases}R_{2m-5}(h\beta)+Q_{m-3}^{-}(h\beta), & \beta<0,\\ F_m(h\beta)\ , & 0\le\beta\le N,\\ -R_{2m-5}(h\beta)+Q_{m-3}^{+}(h\beta), & \beta<0,\end{cases} \tag{12}$$

where

$$\begin{aligned}Q_{m-3}^{-}(h\beta)&=P_{m-3}(h\beta)-Q_{m-3}(h\beta),\\ Q_{m-3}^{+}(h\beta)&=P_{m-3}(h\beta)+Q_{m-3}(h\beta).\end{aligned} \tag{13}$$

$Q_{m-3}^{-}(h\beta)$ va $Q_{m-3}^{+}(h\beta)$ are unknown polynomials of degree $(m-3)$.

Theorem. $C_1[\beta]$, $\beta=1,2,...,N-1$ coefficients of derivative optimal quadrature formula of form (1) for $m\ge 4$ in $L_2^{(m)}(0,1)$ space have the following form [3,4,5]

$$C_1[\beta]=h^3\sum_{k=1}^{m-3}\left(a_k q_k^{\beta}+b_k q_k^{N-\beta}\right),\qquad \beta=1,2,...,N-1 \tag{14}$$

So, $C_1[\beta]$ optimal coefficients depend on $2m-6$ unknowns, and $2m-6$ systems of equations are needed to determine them. First we find from equation (5) the expression of the coefficients $C_1[0]$, $C_1[N]$ for $\alpha=0$ and $\alpha=1$

$$C_1[0]=-\sum_{\beta=1}^{N-1}C_1[\beta](h\beta)-\sum_{\beta=1}^{N-1}C_1[\beta], \tag{15}$$

$$C_1[N]=-\sum_{\beta=1}^{N-1}C_1[\beta](h\beta). \tag{16}$$

It can be seen from equations (15) and (16) that coefficients $C_1[0]$ and $C_1[N]$ are expressed using coefficients $C_1[\beta]$ $(\beta=\overline{1,N-1})$. Therefore, we find the coefficients $C_1[0]$ and $C_1[N]$. For this, we calculate the following sum in the system of equations (3)

$$S=\sum_{\gamma=0}^{N}C_1[\gamma]G_{m-2}\left((h\beta)-(h\gamma)\right)=\sum_{\gamma=0}^{N}C_1[\gamma]\frac{\left|(h\beta)-(h\gamma)\right|^{2m-5}}{2(2m-5)!}=$$

$$=\sum_{\gamma=0}^{\beta}C_1[\gamma]\frac{\left((h\beta)-(h\gamma)\right)^{2m-5}}{(2m-5)!}-\sum_{\gamma=0}^{N}C_1[\gamma]\frac{\left((h\beta)-(h\gamma)\right)^{2m-5}}{2(2m-5)!}=$$

$$=C_1[0]\frac{(h\beta)^{2m-5}}{(2m-5)!}+\sum_{\gamma=1}^{\beta-1}C_1[\gamma]\frac{\left((h\beta)-(h\gamma)\right)^{2m-5}}{(2m-5)!}-\sum_{\gamma=0}^{N}C_1[\gamma]\frac{\left((h\beta)-(h\gamma)\right)^{2m-5}}{2(2m-5)!}=C_1[0]\frac{(h\beta)^{2m-5}}{(2m-5)!}+S_1-S_2 \quad (17)$$

We simplify the sums S_1 and S_2 in the last expression

$$S_1=-\sum_{j=0}^{2m-5}\frac{h^{j+3}(h\beta)^{2m-5-j}}{j!(2m-5-j)!}\left[\sum_{k=1}^{m-3}\frac{a_kq_k}{q_k-1}\sum_{i=0}^{2m-5}\left(\frac{1}{q_k-1}\right)^i\Delta^i0^j+\sum_{k=1}^{m-3}\frac{b_kq_k^N}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i0^j\right] \quad (18)$$

we will simplify the sum of S_2. For this, using the orthogonality condition (5), we obtain the following

$$S_2=\sum_{\gamma=0}^{N}C_1[\gamma]\sum_{j=0}^{2m-5}\frac{(h\beta)^{2m-5-j}(-h\gamma)^j}{2j!(2m-5-j)!}=$$

$$=\sum_{j=m-2}^{2m-5}\frac{(h\beta)^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^j-\sum_{j=0}^{m-3}\frac{(h\beta)^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{i=0}^{j}\frac{j!B_{j+3-i}h^{j+3-i}}{i!(j+3-i)!}. \quad (19)$$

putting equations (18), (19) into (17), we get the following sum

$$S=C_1[0]\frac{(h\beta)^{2m-5}}{(2m-5)!}-\sum_{j=0}^{2m-5}\frac{h^{j+3}(h\beta)^{2m-5-j}}{j!(2m-5-j)!}\left[\sum_{k=1}^{m-3}\frac{a_kq_k}{q_k-1}\sum_{i=0}^{2m-5}\left(\frac{1}{q_k-1}\right)^i\Delta^i0^j+\right.$$

$$\left.+\sum_{k=1}^{m-3}\frac{b_kq_k^N}{1-q_k}\sum_{i=0}^{2m-5}\left(\frac{q_k}{1-q_k}\right)^i\Delta^i0^j\right]-\sum_{j=m-2}^{2m-5}\frac{(h\beta)^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{\gamma=0}^{N}C_1[\gamma](h\gamma)^j+\sum_{j=0}^{m-3}\frac{(h\beta)^{2m-5-j}(-1)^j}{2j!(2m-5-j)!}\sum_{i=0}^{j}\frac{j!B_{j+3-i}h^{j+3-i}}{i!(j+3-i)!}$$

(20)

CONCLUSION

It is known that numerical analytical solutions of differential equations and integral equations formed in mathematical modeling of natural processes are expressed by optimal quadrature and cubature formulas and are the research object of numerical algebra, numerical integration theory and other similar issues. In this regard, it is important to calculate the approximation of exact integrals, as well as to estimate their errors in the Gilbert and Banach spaces of differentiable functions, to construct derivative optimal quadrature formulas. In this work, using the algorithm proposed by Sobolev, a new derived optimal quadrature formula was constructed.

REFERENCES

1. Sobolev S.L. Introduction to the theory of cubature formulas. - M.: Nauka, 1974. - 808 p.
2. Shadimetov, K., Nuraliev, F., Kuziev, S. Optimal quadrature formula of hermite type in the space of differentiable functions // International Journal of Analysis and Applications, 2024, 22, 25, pp. 1-19. https://doi.org/10.28924/2291-8639-22-2024-25.
3. Shadimetov, K., Nuraliev, F., Kuziev, S. Coefficients and Errors of the Optimal Quadrature Formula of the Hermite Type // AIP Conference Proceedings, 2024, 3147(1), 030030, pp. 1-12. https://doi.org/10.1063/5.0210357.
4. Qo'ziyev, S. (2021, April). Methods, tools and forms of distance learning. In *Конференции*.
5. Nuraliev, F. A., & Kuziev, S. S. (2024). Optimal Quadrature Formulas with Derivative in the Space: Optimal Quadrature Formulas with Derivative in the Space. *MODERN PROBLEMS AND PROSPECTS OF APPLIED MATHEMATICS*, *1*(01).
6.